

Multipliers of Hilbert algebras and deformation quantization*

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Abstract

In this paper, we introduce the notion of multiplier of a Hilbert algebra. The space of bounded multipliers is a semifinite von Neumann algebra isomorphic to the left von Neumann algebra of the Hilbert algebra, as expected. However, in the unbounded setting, the space of multipliers has the structure of a $*$ -algebra with nice properties concerning commutant and affiliation: it is a pre-GW $*$ -algebra. And this correspondence between Hilbert algebras and its multipliers is functorial. Then, we can endow the Hilbert algebra with a nice topology constructed from unbounded multipliers. As we can see from the theory developed here, multipliers should be an important tool for the study of unbounded operator algebras.

We also formalize the remark that examples of non-formal deformation quantizations give rise to Hilbert algebras, by defining the concept of Hilbert deformation quantization (HDQ) and studying these deformations as well as their bounded and unbounded multipliers in a general way. Then, we reformulate the notion of covariance of a star-product in this framework of HDQ and multipliers, and we call it a symmetry of the HDQ. By using the multiplier topology of a symmetry, we are able to produce various functional spaces attached to the deformation quantization, like the generalization of Schwartz space, Sobolev spaces, Gracia-Bondia-Varilly spaces. Moreover, the non-formal star-exponential of the symmetry can be defined in full generality and has nice relations with these functional spaces. We apply this formalism to the Moyal-Weyl deformation quantization and to the deformation quantization of Kählerian Lie groups with negative curvature.

Key Words: Multiplier, Hilbert algebra, unbounded operator, deformation quantization, symmetry, Fréchet algebra, distribution, star-exponential.

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1 Introduction

1.1 Motivations

The theory of operator algebras (see [1, 2]), C*-algebras and von Neumann algebras, was initiated by some mathematicians in the context of quantum mechanics during the 40ies and in particular by J. von Neumann and his celebrated bicommutant theorem [3]. Briefly speaking, a von Neumann algebra is a *-algebra of bounded operators on a Hilbert space with identity operator and weakly closed. Later, Dixmier introduced Hilbert algebras for the classification of semifinite von Neumann algebras and to show the commutant theorem [4]. A Hilbert algebra, endowed with a product, an involution and a scalar product, gives rise to a semifinite von Neumann algebra (its left or right von Neumann algebra) and any semifinite von Neumann algebra can be obtained in this way (up to isomorphism). This structure was then extended to the purely

infinite case (see [1]). The von Neumann algebras have nowadays applications in numerous domains of mathematics and physics, like knot theory, representation theory, noncommutative geometry, logics, probability, quantum field theory, statistical mechanics...

On another side, multipliers were introduced for C^* -algebras in the 60ies [5] and then extended to various contexts as Fréchet algebras. This tool showed its importance for the theory of C^* -algebras and in particular for locally compact quantum groups. A natural question relies on what multipliers of a Hilbert algebra should be. In a not so surprisingly way, such multipliers form a von Neumann algebra isomorphic to the left or right von Neumann algebra of the given Hilbert algebra. But this is not the end point concerning this structure...

Indeed, operators appearing naturally in quantum mechanics are often unbounded as position and momentum operators. Then, mathematicians started to study $*$ -algebras of unbounded operators with a common dense subdomain [6, 7] called later O^* -algebras (see [8, 9] for a review on the subject). Properties of von Neumann algebras, generalized in appropriate ways to unbounded operator algebras, lead to the notions of EW^* -algebras (symmetric O^* -algebra \mathbf{M} with bounded part $\bar{\mathbf{M}}_b$ von Neumann)[10] and GW^* -algebras (O^* -algebra equal to is bicommutant, with commutant stabilizing the domain, and one additional topological constraint on the domain) [11].

Then, the question concerning multipliers of Hilbert algebras can be asked in the unbounded context. Can we define a notion of unbounded multiplier of a Hilbert algebra? Does this notion fit in the generalizations of von Neumann algebras that are EW^* - or GW^* -algebras? We will see in this paper that such multipliers $\mathfrak{M}(\mathbf{A})$ of a Hilbert algebra \mathbf{A} exist but they do not form an EW^* -algebra in general, even not a GW^* -algebra, but only a pre- GW^* -algebra, relaxing the topological constraint on the domain. Moreover, one can associate to $\mathfrak{M}(\mathbf{A})$ a GW^* -algebra $\mathfrak{M}_{\text{topo}}(\mathbf{A})$ giving rise to an interesting topology on the domain \mathbf{A} . As it is the case for (bounded) von Neumann algebras, such unbounded multipliers of Hilbert algebras should probably be crucial tools for the study of unbounded operator algebras.

Deformation quantization yields interesting examples of operator algebras related to Poisson geometry and it was also introduced in the context of quantum mechanics [12]. A deformation quantization of Poisson manifold M is the data of an associative product \star_θ on $C^\infty(M)$ (or one of its dense subspace) depending on a deformation parameter θ , constructed from the Poisson structure, and such that it corresponds to the commutative pointwise product for $\theta = 0$.

On one side, one can consider formal deformation quantization, where the star-product $f_1 \star_\theta f_2 = \sum_{k=0}^{\infty} C_k(f_1, f_2) \theta^k \in C^\infty(M)[[\theta]]$ ($f_i \in C^\infty(M)$) is a formal power series in the deformation parameter θ . Such deformation quantizations were intensively studied, and definitely classified in [13]. For example, there is only one class of deformation on $M = \mathbb{R}^{2n}$ given by the Moyal product

$$f_1 \star_\theta f_2(x) = \sum_{k=0}^{\infty} \left(\frac{-i\theta}{2} \right)^k \omega(\partial_x, \partial_y)^k f_1(x) f_2(y)|_{y=x} \quad (1.1)$$

for $f_i \in C^\infty(\mathbb{R}^{2n})$ and ω a symplectic structure on \mathbb{R}^{2n} . In the more general case where $G = M$ is a Lie group, such a deformation then corresponds to a Drinfeld twist that permits to deform also algebras on which G acts [14].

In a non-formal point of view, more interesting for functional analysis, and where the deformation parameter θ has now a real value, there is no classification of deformation quantizations, and there are actually only few available examples. For the Abelian group $G = \mathbb{R}^{2n}$, the star-product (1.1) acquires a non-formal meaning in the Moyal-Weyl formula

$$f_1 \star_\theta f_2(x) = \frac{1}{(\pi\theta)^{2n}} \int f_1(y) f_2(z) e^{-\frac{2i}{\theta}(\omega(y,z) + \omega(z,x) + \omega(x,y))} dy dz \quad (1.2)$$

for $f_i \in \mathcal{S}(\mathbb{R}^{2n})$, and for which (1.1) is the asymptotic expansion in θ close to 0. Rieffel then used this Abelian symmetry and this deformation in order to build a non-formal twist, also called universal deformation formula (UDF), that deforms continuously the C^* -algebras on which \mathbb{R}^{2n} is acting [15]. The star-exponential associated to the star-product (1.2) has been well-defined in a non-formal way and produces various applications in harmonic analysis [16, 17, 18]. Deformation quantizations of Abelian symmetries were also extended to the complex case [19, 20], to the case of supergroups $\mathbb{R}^{2n|m}$ [21] and to Abelian p -adic groups [22].

In order to construct deformation quantizations of non-Abelian groups, Bieliavsky developed a retract method also based on symmetries [23, 24, 25]. Starting from a non-formal star-product \star_θ^1 on M , which is G_1 -strongly invariant and G_2 -covariant, if the shared symmetry H contained in both G_1 and G_2 is “sufficiently large”, the retract method then constructs another non-formal star-product \star_θ^2 on M that is G_2 -strongly invariant, as well as the explicit non-formal intertwiner U_θ between \star_θ^1 and \star_θ^2 . This method was successfully applied to find non-formal deformation quantizations \star_θ^2 of negatively curved Kählerian Lie groups, also called normal j -groups in [26], starting from the Moyal-Weyl product \star_θ^1 [24, 25]. An associated pseudodifferential calculus and a UDF were also built in [25], and the corresponding star-exponential was exhibited in [27] with applications in harmonic analysis of these Lie groups.

It turns out [15] that the space $L^2(\mathbb{R}^{2n})$ endowed with the Moyal-Weyl product \star_θ^1 , the complex conjugation and its standard scalar product is a Hilbert algebra. In the same way, $L^2(M)$ with the product \star_θ^2 in the case of normal j -groups is [25] also a Hilbert algebra, while the intertwiner U_θ can be viewed as a unitary $*$ -homomorphism. Therefore this motivates the introduction of the notion of Hilbert deformation quantization (HDQ) in this paper, just formalizing these observations. As a consequence, bounded and unbounded multipliers can play a role for HDQ. Moreover, the symmetries of HDQ will allow defining in full generality various functional spaces adapted to these deformations. For example, such functional spaces in the case of $M = \mathbb{R}^{2n}$ were very useful to define spectral triple on the Moyal-Weyl deformation quantization [28]. Furthermore, the non-formal star-exponential can also be defined in a general way by using multipliers and has interesting relations with these functional spaces. All these applications stress that the context of Hilbert algebras and its bounded and unbounded multipliers is very adapted to non-formal deformation quantization, as well as symmetries. Indeed, this crucial concept of symmetry, already producing deformations by retract method and by Drinfeld twists, gives rise now directly to the topology of functional spaces adapted to these deformations.

Eventually, notice that a non-formal $SL(2, \mathbb{R})$ -strongly invariant deformation quantization was recently exhibited [29], and one can see that it gives also rise to a HDQ. Then, the formalism developed in this paper can be applied for this HDQ and for the symmetry $SL(2, \mathbb{R})$, as for other more general symmetries and deformations. Explicit computation of the non-formal star-exponential associated to this symmetry $SL(2, \mathbb{R})$ can be found in [30] as well as its link with multipliers.

1.2 Outline of the paper

In section 2.1, we recall the well-known setting of Hilbert algebras - algebras with involution and scalar product - introduced by Dixmier, and their links with semifinite von Neumann algebras. Then, we present the framework of unbounded operator algebras in section 2.2. An O^* -algebra on the dense domain \mathcal{D} is an algebra of unbounded operators whose domains contain \mathcal{D} , which stabilize \mathcal{D} , as well as their adjoints. Such an O^* -algebra induces on \mathcal{D} its graphic topology. Bounded and unbounded commutants of O^* -algebras are also recalled in section 2.3, which give rise to the notion of (pre-) GW^* -algebra, an extension of von Neumann algebras to the

unbounded setting.

In section 3.1, we start by defining bounded $\mathfrak{M}_b(\mathbf{A})$ and unbounded $\mathfrak{M}(\mathbf{A})$ multipliers of a given Hilbert algebra \mathbf{A} , inspired directly by the definition of double-centralizers of C^* -algebras. We show in section 3.2 that bounded multipliers form a von Neumann algebra isomorphic to the left or right von Neumann algebra, as expected. For the space of unbounded multipliers $\mathfrak{M}(\mathbf{A})$, we prove in section 3.3 that it has a structure of O^* -algebra, we characterize its bounded commutant and its unbounded bicommutant, and we arrive to the fact that it is a pre-GW*-algebra (non-closed in general). It turns out as seen in section 3.4 that this multiplier construction defines a covariant functor between Hilbert algebras with isomorphisms as arrows and the category of von Neumann algebras or pre-GW*-algebras with spatial isomorphisms as arrows.

We consider in section 3.5 natural topologies on the Hilbert algebra \mathbf{A} associated to multipliers. It turns out that the graphic topology of $\mathfrak{M}(\mathbf{A})$ is not interesting here, but we can associate to $\mathfrak{M}(\mathbf{A})$ another O^* -algebra $\mathfrak{M}_{\text{topo}}(\mathbf{A})$, whose graphic topology $\tau_{\mathfrak{M}}$ on \mathbf{A} is called the multiplier topology; and this topology will have interesting applications to deformation quantization. We also endow $\mathfrak{M}(\mathbf{A})$ with a natural locally convex complete topology, and we look in section 3.6 at some particular examples as multipliers of unital or commutative Hilbert algebras.

In section 4.1, we first define in a general way the concept of Hilbert deformation quantization (HDQ), which is just a non-formal deformation quantization with a Hilbert algebra structure, and we give some basic properties concerning its multipliers. For a given HDQ \mathbf{A}_θ , we then introduce in section 4.2 the crucial notion of symmetry \mathfrak{g} , which expresses the covariance of the deformation quantization in the language of HDQ. We show that a symmetry \mathfrak{g} of a HDQ \mathbf{A}_θ forms a subalgebra of multipliers whose multiplier topology generates a Fréchet algebra, Hilbert subalgebra of \mathbf{A}_θ , denoted by $\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g})$ in analogy of the Schwartz space. Moreover, any representation of \mathbf{A}_θ with some natural conditions, i.e. quantization map associated to the deformation quantization, can be extended to unbounded multipliers $\mathfrak{M}(\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g}))$.

In this operator setting of HDQ, we also define in a general way the star-exponential of a symmetry \mathfrak{g} in section 4.3, which is a unitary multiplier satisfying the Baker-Campbell-Hausdorff property. Furthermore, other interesting functional spaces, $\mathbf{G}^{kl}(\mathbf{A}_\theta, \mathfrak{g})$ generalizing topological spaces of Gracia-Bondia-Varilly, and $\mathbf{H}^k(\mathbf{A}_\theta, \mathfrak{g})$ generalizing Sobolev spaces, are also associated to any symmetry \mathfrak{g} in section 4.4. In the case of an invariant symmetry (the deformation \star_θ is strongly invariant), all these functional spaces can be characterized as smooth vectors for group actions in section 4.5.

In section 5.1, we look at the well-known example of Moyal-Weyl deformation quantization that defines a HDQ \mathbf{A}_θ , and for which the Weyl quantization is a representation. The translation group induces an invariant symmetry \mathfrak{g} of this HDQ \mathbf{A}_θ . For this basic example, we find for $\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g})$ the usual Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, for $\mathbf{G}^{kl}(\mathbf{A}_\theta, \mathfrak{g})$ the topological spaces introduced by Gracia-Bondia-Varilly with the matrix basis (see section 5.2), for $\mathbf{H}^k(\mathbf{A}_\theta, \mathfrak{g})$ the usual Sobolev spaces $H^k(\mathbb{R}^{2n})$, and for the star-exponential the usual exponential. And we show in section 5.3 that the unbounded multipliers $\mathfrak{M}(\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g}))$ corresponds to the usual \star_θ -multipliers of the Fréchet algebra $\mathcal{S}(\mathbb{R}^{2n})$. To illustrate the unital case, we also mention in section 5.4 the infinite-dimensional Clifford algebras, seen as limit of HDQs.

Finally, we consider in section 5.5 the non-formal deformation quantization of Kählerian Lie groups with negative curvature, which is also a HDQ \mathbf{A}_θ with a representation. The bounded symmetric domains associated to these groups, and on which the HDQ lives, possess an interesting transvection group. Such a transvection group indeed induces an invariant symmetry of the HDQ \mathbf{A}_θ (bigger than the one induced by the Kählerian group itself). A theorem in section 5.6 shows that $\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g})$ is identical to the modified Schwartz space introduced by Bieliavsky-Gayral for this deformation. The star-exponential for this symmetry can also be obtained explicitly by

using computations of a previous paper. We conclude by expressing in a explicit way the new functional spaces $H^k(\mathbf{A}_\theta, \mathfrak{g})$ generated by this symmetry and adapted to this star-product.

2 Operator Algebra framework

To fix notations and conventions and to be self-contained, we recall the basics of Hilbert algebras theory (see [4] and references therein, where all the proofs are given) and the theory of O^* -algebras and GW^* -algebras (see [8, 9]) that will be useful in this paper.

2.1 Hilbert algebras

We just recall here what are Hilbert algebras and give some of their fundamental properties.

Definition 2.1 Let \mathbf{A} be an algebra over \mathbb{C} with an involution and a scalar product¹. We say that \mathbf{A} is a **Hilbert algebra** if

1. for any $x, y \in \mathbf{A}$, $\langle y^*, x^* \rangle = \langle x, y \rangle$,
2. for any $x, y, z \in \mathbf{A}$, $\langle xy, z \rangle = \langle y, x^* z \rangle$,
3. for any $x \in \mathbf{A}$, the map $\lambda_x : y \in \mathbf{A} \mapsto xy$ is continuous,
4. the set $\{xy, x, y \in \mathbf{A}\}$ is dense in \mathbf{A} .

We note $\mathcal{H}_{\mathbf{A}}$ the Hilbert space that is the completion of \mathbf{A} for the norm $\|x\| := \sqrt{\langle x, x \rangle}$ associated to the scalar product. \blacklozenge

For $x \in \mathbf{A}$, it turns out that $\rho_x : y \in \mathbf{A} \mapsto yx$ is also continuous. The maps λ, ρ extend to a $*$ -algebra morphism $\lambda : \mathbf{A} \rightarrow \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ and a $*$ -algebra antimorphism $\rho : \mathbf{A} \rightarrow \mathcal{B}(\mathcal{H}_{\mathbf{A}})$, where $\mathcal{B}(\mathcal{H})$ denotes the bounded operators on the Hilbert space \mathcal{H} . We note $\lambda(\mathbf{A})$ (resp. $\rho(\mathbf{A})$) the weak closure of the image of \mathbf{A} by λ (resp. ρ) and they are called **left** (resp. **right**) **von Neumann algebras** of the Hilbert algebra \mathbf{A} . They satisfy $\lambda(\mathbf{A})' = \rho(\mathbf{A})$.

The involution extends continuously to a continuous operator on $\mathcal{H}_{\mathbf{A}}$. Moreover, we can show that $\|x^*\| = \|x\|$ for all $x \in \mathcal{H}_{\mathbf{A}}$, but the norm do not satisfy in general the C^* -property. The space of Hilbert-Schmidt operators $\mathcal{L}^2(\mathcal{H})$ of a Hilbert space \mathcal{H} is an example of Hilbert algebra.

Definition 2.2 An element $x \in \mathcal{H}_{\mathbf{A}}$ is called **bounded** if there exists $\lambda_x \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ such that $\forall y \in \mathbf{A}$, $\lambda_x(y) = \rho_y(x)$, or equivalently if there exists $\rho_x \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ such that $\forall y \in \mathbf{A}$, $\rho_x(y) = \lambda_y(x)$. We note \mathbf{A}_b the set of all bounded elements in $\mathcal{H}_{\mathbf{A}}$, also called the **fulfilment** of \mathbf{A} .

A Hilbert algebra is called **full** if it contains all the bounded elements, i.e. $\mathbf{A} = \mathbf{A}_b$. \blacklozenge

For any Hilbert algebra \mathbf{A} , \mathbf{A}_b is also a Hilbert algebra (with same Hilbert completion) and by using λ, ρ , \mathbf{A}_b is included in $\lambda(\mathbf{A})$ and $\rho(\mathbf{A})$.

Theorem 2.3 Let \mathbf{A} be a Hilbert algebra. For $S \in \lambda(\mathbf{A})^+$ (resp. $T \in \rho(\mathbf{A})^+$), we define

$$\tau_\lambda(S) := \langle x, x \rangle, \quad \tau_\rho(T) := \langle y, y \rangle,$$

if $S^{\frac{1}{2}} = \lambda_x$ (resp. $T^{\frac{1}{2}} = \rho_y$) for some x (resp. y) bounded in $\mathcal{H}_{\mathbf{A}}$; and $\tau_\lambda(S) := +\infty$ (resp. $\tau_\rho(T) := +\infty$) otherwise. Then, τ_λ (resp. τ_ρ) is a faithful semifinite normal trace on $\lambda(\mathbf{A})^+$ (resp. $\rho(\mathbf{A})^+$). And $\tau_\lambda(\lambda_x^* \lambda_y) = \langle x, y \rangle$ for any $x, y \in \mathbf{A}_b$.

¹We choose the convention that the scalar product $\langle -, - \rangle$ is left antilinear and right linear.

The traces τ_λ and τ_ρ are called the **natural traces** on $\lambda(\mathbf{A})^+$ and $\rho(\mathbf{A})^+$. It turns out that any von Neumann algebra with semifinite faithful normal trace is isomorphic to the left or right von Neumann algebra of a Hilbert algebra.

Corollary 2.4 *Let \mathbf{A} be a Hilbert algebra. If $x \in \mathbf{A}_b$ bounded satisfies $\rho_y(x) = 0$ for all $y \in \mathbf{A}$, then $x = 0$.*

PROOF Suppose that $\rho_y(x) = 0$ for any $y \in \mathbf{A}$. Then, it can be extended for any $y \in \mathbf{A}_b$. Take $y = x^*$, we have $\lambda_x \lambda_{x^*} = 0$ and we can apply the trace: $0 = \tau_\lambda(\lambda_x \lambda_{x^*}) = \langle x^*, x^* \rangle$. Then $x = 0$. ■

Proposition 2.5 *Let \mathbf{A}_1 and \mathbf{A}_2 be Hilbert algebras. On the algebraic direct sum $\mathbf{A}_1 \oplus \mathbf{A}_2$, one can define the following natural structures:*

- *Vector space:* $(a_1, a_2) + \lambda(b_1, b_2) := (a_1 + \lambda b_1, a_2 + \lambda b_2)$,
- *Product:* $(a_1, a_2)(b_1, b_2) := (a_1 b_1, a_2 b_2)$,
- *Involution:* $(a_1, a_2)^* := (a_1^*, a_2^*)$,
- *Scalar product:* $\langle (a_1, a_2), (b_1, b_2) \rangle := \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle$,

for $a_i, b_i \in \mathbf{A}_i$. Then, $\mathbf{A}_1 \oplus \mathbf{A}_2$ is a Hilbert algebra and its completion is $\mathcal{H}_{\mathbf{A}_1} \hat{\oplus} \mathcal{H}_{\mathbf{A}_2}$.

Proposition 2.6 *Let \mathbf{A} and \mathbf{B} be two Hilbert algebras. Then the natural structure of the algebraic tensor product are $\mathbf{A} \otimes \mathbf{B}$: $\forall a_i \in \mathbf{A}, \forall b_i \in \mathbf{B}$,*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (a_1 a_2) \otimes (b_1 b_2), \quad (a \otimes b)^* := a^* \otimes b^*, \quad \langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle.$$

Then, $\mathbf{A} \otimes \mathbf{B}$ is a Hilbert algebra and $\mathcal{H}_{\mathbf{A} \otimes \mathbf{B}} = \mathcal{H}_{\mathbf{A}} \hat{\otimes} \mathcal{H}_{\mathbf{B}}$ (completed tensor product with respect to the scalar product).

Definition 2.7 Let $Z_{\mathbf{A}} = \{z \in \mathbf{A}_b, \lambda_z = \rho_z\}$ be the **bounded center** relative to the Hilbert algebra \mathbf{A} . ♦

This is a subspace of $\mathcal{H}_{\mathbf{A}}$ depending only on \mathbf{A}_b , not on the choice of the Hilbert algebra \mathbf{A} [4].

2.2 O*-algebras

In general, unbounded operators cannot be composed and they do not form an algebra. However, if the operators considered have a common dense subdomain and if they stabilize this subdomain, then composition is well-defined and one obtains an algebra. By adding the same conditions for the adjoint in order to get a *-algebra, we arrive to the following definitions.

Let \mathcal{H} be a Hilbert space and \mathcal{D} be a dense subspace of \mathcal{H} . Let also

$$\mathcal{L}^+(\mathcal{D}) := \{T : \mathcal{D} \rightarrow \mathcal{D} \text{ linear with } \mathcal{D} \subset \text{Dom}(T^*) \text{ and } T^*(\mathcal{D}) \subset \mathcal{D}\},$$

where $\text{Dom}(T)$ denotes the domain of the operator T . It is a *-algebra of (closable) unbounded operators with the usual composition of operators and the involution given by the restriction of the adjoint to \mathcal{D} : $T^+ := (T^*)|_{\mathcal{D}}$. The identity on \mathcal{D} will be denoted by $\mathbb{1}$ and belongs to $\mathcal{L}^+(\mathcal{D})$.

Definition 2.8 An **O*-algebra** of domain \mathcal{D} is a *-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing $\mathbb{1}$. ♦

If \mathbf{M} is an O*-algebra of domain \mathcal{D} , we denote by \mathbf{M}_b its bounded part, i.e. the bounded operators T of $\mathcal{B}(\mathcal{H})$ such that $T|_{\mathcal{D}} \in \mathbf{M}$.

Definition 2.9 Let \mathbf{M} be an O^* -algebra of domain \mathcal{D} . The **graphic topology** of \mathcal{D} is a locally convex one associated to the seminorms $\|x\|_T := \|T(x)\|$, for $T \in \mathbf{M}$, $x \in \mathcal{D}$ and $\|x\| = \sqrt{\langle x, x \rangle}$. ♦

This topology denoted by $\tau_{\mathbf{M}}$ is stronger than the one induced by the Hilbert space \mathcal{H} . We note τ_+ the graphic topology on \mathcal{D} associated to the O^* -algebra $\mathcal{L}^+(\mathcal{D})$. Let $\mathcal{L}^+(\mathcal{D}, \tau_{\mathbf{M}})$ be the space of operators $T \in \mathcal{L}^+(\mathcal{D})$ such that T and T^+ are continuous for the topology $\tau_{\mathbf{M}}$. Then, it is an O^* -algebra containing \mathbf{M} , and the graphic topology on \mathcal{D} associated to it coincide with $\tau_{\mathbf{M}}$. Note also that if $\tau_{\mathbf{M}} = \tau_+$, which is the case if $(\mathcal{D}, \tau_{\mathbf{M}})$ is a Fréchet space, then $\mathcal{L}^+(\mathcal{D}, \tau_{\mathbf{M}}) = \mathcal{L}^+(\mathcal{D})$.

Definition 2.10 An O^* -algebra \mathbf{M} on the domain \mathcal{D} is said to be **closed** if $(\mathcal{D}, \tau_{\mathbf{M}})$ is complete. ♦

If \mathbf{M} is not closed, then $\tilde{\mathcal{D}} := \bigcap_{T \in \mathbf{M}} \text{Dom}(\overline{T})$ is the completion of \mathcal{D} , and $\tilde{\mathbf{M}} := \{(\overline{T})|_{\tilde{\mathcal{D}}}, T \in \mathbf{M}\}$ is a closed O^* -algebra, called the **closure** of \mathbf{M} . Be care of the fact that $\tilde{\mathcal{D}} \subsetneq \bigcap_{T \in \mathbf{M}} \text{Dom}(T^*)$ in general.

2.3 GW*-algebras

Definition 2.11 The **weak bounded commutant** of an O^* -algebra \mathbf{M} is defined as

$$\mathbf{M}'_w := \{T \in \mathcal{B}(\mathcal{H}) \mid \forall x, y \in \mathcal{D}, \forall S \in \mathbf{M}, \langle y, TS(x) \rangle = \langle S^+(y), T(x) \rangle\}.$$

It is a weakly closed $*$ -invariant subspace, but not an algebra in general. ♦

There exist other notions of bounded (or unbounded) commutants than the ones presented here but we don't need them in the following.

Definition 2.12 Let T be a closed operator on \mathcal{H} and \mathbf{N} a von Neumann algebra on \mathcal{H} . We recall that T is said **affiliated** with \mathbf{N} if T commutes with all operators in \mathbf{N}' , i.e. if $\forall S \in \mathbf{N}'$, $ST \subset TS$, that is $\forall x \in \text{Dom}(T)$, $S(x) \in \text{Dom}(T)$ and $TS(x) = S(T(x))$. ♦

Proposition 2.13 Let \mathbf{M} be an O^* -algebra on \mathcal{D} . Then $\mathbf{M}'_w(\mathcal{D}) \subset \tilde{\mathcal{D}}$ if and only if \overline{T} is affiliated with $(\mathbf{M}'_w)'$ for any $T \in \mathbf{M}$. If this is satisfied, then \mathbf{M}'_w is a von Neumann algebra.

The usual weak, strong and strong* topologies can also be defined on $\mathcal{L}^+(\mathcal{D})$. They are respectively defined by the seminorms $\langle x, T(y) \rangle$, $\|T(x)\|$, $\|T(x)\| + \|T^+(x)\|$, for $T \in \mathcal{L}^+(\mathcal{D})$ and $x, y \in \mathcal{D}$, and denoted by τ_w , τ_s and τ_{s*} . We denote by $[\mathbf{M}]^{s*}$ the closure of $\mathbf{M} \subset \mathcal{L}^+(\mathcal{D})$ in $\mathcal{L}^+(\mathcal{D})$ for the topology τ_{s*} .

Definition 2.14 The **weak unbounded commutant** of an O^* -algebra \mathbf{M} is defined as

$$\mathbf{M}'_c := \{T \in \mathcal{L}^+(\mathcal{D}) \mid \forall x, y \in \mathcal{D}, \forall S \in \mathbf{M}, \langle S^+(y), T(x) \rangle = \langle T^+(y), S(x) \rangle\}.$$

It is an O^* -algebra on \mathcal{D} . The **unbounded bicommutant** of \mathbf{M} is

$$\mathbf{M}''_{wc} := (\mathbf{M}'_w)'_c$$

and it is an O^* -algebra τ_{s*} -closed (in $\mathcal{L}^+(\mathcal{D})$), of weak bounded commutant $(\mathbf{M}''_{wc})'_w = \mathbf{M}'_w$. ♦

Remark 2.15 If \mathbf{M} is an O^* -algebra such that \mathbf{M}'_w is a von Neumann algebra, then $\mathbf{M}''_{wc} = [(\mathbf{M}'_w)'_{\mathcal{D}}]^{s*}$. ♦

Definition 2.16 Let \mathbf{M} be an O^* -algebra. If $\mathbf{M}'_w(\mathcal{D}) \subset \tilde{\mathcal{D}}$ and $\mathbf{M} = \mathbf{M}''_{wc}$, \mathbf{M} is called a **pre-GW*-algebra**. If moreover \mathbf{M} is closed ($\tilde{\mathcal{D}} = \mathcal{D}$), \mathbf{M} is called a **GW*-algebra**. ♦

Proposition 2.17 *If \mathbf{M} is a pre-GW*-algebra on \mathcal{D} , then its closure $\tilde{\mathbf{M}}$ is a GW*-algebra on $\tilde{\mathcal{D}}$ and*

$$\mathbf{M}'_w = (\tilde{\mathbf{M}})'_w, \quad \mathbf{M}''_{wc} = [(\mathbf{M}'_w)']_{\mathcal{L}^+(\mathcal{D})}^{s*}.$$

Proposition 2.18 *Let \mathbf{M} be a closed O^* -algebra. Then, \mathbf{M} is a GW*-algebra if and only if there exists a von Neumann algebra \mathbf{N} on \mathcal{H} such that $\mathbf{N}'(\mathcal{D}) \subset \mathcal{D}$ and $\mathbf{M} = [\mathbf{N}_{|\mathcal{D}}]^{s*}$. In this case, $(\mathbf{M}'_c)'_c = \mathbf{M}$ and $\mathbf{N} = (\mathbf{M}'_w)'$.*

Example 2.19 Let \mathcal{D} be a dense subspace of \mathcal{H} such that $\mathcal{L}^+(\mathcal{D})$ is closed. Then, $\mathcal{L}^+(\mathcal{D})$ is a GW*-algebra on \mathcal{D} , its weak bounded commutant is $\mathcal{L}^+(\mathcal{D})'_w = \mathbb{C}\mathbb{1}$ so that its bounded bicommutant is $(\mathcal{L}^+(\mathcal{D})'_w)' = \mathcal{B}(\mathcal{H})$. However, note that $(\mathcal{L}^+(\mathcal{D})'_w)' \neq \mathcal{L}^+(\mathcal{D})_b$. \blacklozenge

3 Theory of multipliers

3.1 Definition

We consider here a Hilbert algebra \mathbf{A} and we use the notations introduced in section 2.1. Then, we can adapt the definition of multiplier of a C*-algebra or a Fréchet algebra to this context of Hilbert algebras in two different ways: one within the context of unbounded operators leaving \mathbf{A} invariant and the other within the context of bounded operators.

Definition 3.1 We define an **unbounded multiplier** (also called simply a multiplier) of the Hilbert algebra \mathbf{A} to be a pair $T = (L, R)$ of operators $L, R \in \mathcal{L}^+(\mathbf{A})$ such that

$$\forall x, y \in \mathbf{A} \quad : \quad xL(y) = R(x)y,$$

where $\mathcal{L}^+(\mathbf{A})$ has been given in section 2.3. Let us denote by $\mathfrak{M}(\mathbf{A})$ the set of all (unbounded) multipliers of \mathbf{A} . This is a subset of $\mathcal{L}^+(\mathbf{A} \oplus \mathbf{A})$. \blacklozenge

Definition 3.2 On another side, we define a **bounded multiplier** of \mathbf{A} to be a pair $T = (L, R) \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ such that

$$\forall x, y \in \mathbf{A} \quad : \quad \lambda_x L(y) = \rho_y R(x).$$

Note that we didn't write $xL(y) = R(x)y$ as before because $L(y)$ and $R(x)$ do not belong to \mathbf{A} a priori. We denote by $\mathfrak{M}_b(\mathbf{A})$ the set of all bounded multipliers of \mathbf{A} . This is a subset of $\mathcal{B}(\mathcal{H}_{\mathbf{A}} \oplus \mathcal{H}_{\mathbf{A}})$. \blacklozenge

Lemma 3.3 *If $x \in \mathcal{H}_{\mathbf{A}}$ is bounded, i.e. $x \in \mathbf{A}_b$, then for any $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})$, $L(x)$ and $R(x)$ are in \mathbf{A}_b .*

PROOF First if $x \in \mathbf{A}$, Definition 3.2 gives that $L(x)$ and $R(x)$ are bounded and that $\lambda_{R(x)} = \lambda_x L$, $\rho_{L(x)} = \rho_x R$. Then we have $\rho_{L(y)}(x) = \rho_y R(x)$ for any $x, y \in \mathbf{A}$, and this extends to any $x \in \mathbf{A}_b$ by density. This means that $\lambda_x L(y) = \rho_{L(y)}(x) = \rho_y R(x)$ for $x \in \mathbf{A}_b$ and $y \in \mathbf{A}$. Thus $R(x)$ is bounded and $\lambda_{R(x)} = \lambda_x L$. We can do the same thing for $L(x)$. \blacksquare

This means that \mathbf{A}_b is a *-algebra stable under bounded multipliers. Actually, it turns out that if \mathbf{B} is dense Hilbert subalgebra of \mathbf{A} , $\mathfrak{M}_b(\mathbf{B}) = \mathfrak{M}_b(\mathbf{A})$, so that the bounded multipliers depend only of the data of the algebra of bounded elements.

Definition 3.4 • A **left multiplier** (resp. **right multiplier**) of \mathbf{A} is an operator $L \in \mathcal{L}^+(\mathbf{A})$ (resp. $R \in \mathcal{L}^+(\mathbf{A})$) satisfying

$$\forall x, y \in \mathbf{A} \quad : \quad L(xy) = L(x)y \quad (\text{resp. } R(xy) = xR(y)).$$

We note $L \in \mathfrak{M}_L(\mathbf{A})$ (resp. $R \in \mathfrak{M}_R(\mathbf{A})$).

• A **left bounded multiplier** (resp. **right bounded multiplier**) of \mathbf{A} is an operator $L \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ (resp. $R \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$) satisfying

$$\forall x, y \in \mathbf{A} \quad : \quad L(xy) = \rho_y L(x) \quad (\text{resp. } R(xy) = \lambda_x R(y)).$$

We note $L \in \mathfrak{M}_{L,b}(\mathbf{A})$ (resp. $R \in \mathfrak{M}_{R,b}(\mathbf{A})$). ◆

So, (bounded or unbounded) left (resp. right) multipliers are right (resp. left) \mathbf{A} -module homomorphisms. Let us see their relation with multipliers of Definitions 3.1 and 3.2. Denote by $J : x \mapsto x^* \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ the involution of \mathbf{A} extended to $\mathcal{H}_{\mathbf{A}}$.

Proposition 3.5 *Let $L, R \in \mathcal{B}(\mathcal{H}_{\mathbf{A}})$. We have the equivalence between*

1. (L, R) is a bounded multiplier in $\mathfrak{M}_b(\mathbf{A})$.
2. L is in $\mathfrak{M}_{L,b}(\mathbf{A})$ and $R = JL^*J$.
3. R is in $\mathfrak{M}_{R,b}(\mathbf{A})$ and $L = JR^*J$.

PROOF • $1 \Rightarrow 2$: let $(L, R) \in \mathfrak{M}_b(\mathbf{A})$. Then $\forall x, y, z \in \mathbf{A}$, $\lambda_x L(yz) = \rho_{yz} R(x) = \rho_z \lambda_x L(y) = \lambda_x \rho_z L(y)$. Corollary 2.4 and Lemma 3.3 imply that $L \in \mathfrak{M}_{L,b}(\mathbf{A})$. Next for $x, y, z \in \mathbf{A}$,

$$\begin{aligned} \langle R(x^*z)^*, y \rangle &= \langle (\lambda_x^* R(z))^*, y \rangle = \langle \rho_x R(z)^*, y \rangle = \langle x, \rho_y R(z) \rangle \\ &= \langle x, \lambda_z L(y) \rangle = \langle z^*x, L(y) \rangle = \langle L^*(z^*x), y \rangle, \end{aligned}$$

by using standard properties $(\lambda_x(z))^* = \rho_x^*(z^*)$ and $\langle \rho_x(z), y \rangle = \langle x, \rho_y(z^*) \rangle$, for $x, y \in \mathbf{A}$ and $z \in \mathbf{A}_b$. This gives $L^*(z^*x) = R(x^*z)^*$. By using the fact that $L^*(xy) = \rho_y L^*(x)$ thanks to

$$\langle L^*(xy), z \rangle = \langle xy, L(z) \rangle = \langle x, \rho_y^* L(z) \rangle = \langle x, L(zy^*) \rangle = \langle \rho_y L^*(x), z \rangle,$$

we deduce that $R = JL^*J$.

• For the reciproc $2 \Rightarrow 1$, we suppose that $L \in \mathfrak{M}_{L,b}(\mathbf{A})$ and $R = JL^*J$. Then for $x, y, z \in \mathbf{A}$,

$$\langle \lambda_y^* L^*(x), z \rangle = \langle L^*(x), yz \rangle = \langle x, L(yz) \rangle = \langle x, \rho_z L(y) \rangle = \langle \rho_x(L(y)^*), z \rangle,$$

so $\lambda_y^* L^*(x) = \rho_x(L(y)^*)$. As a consequence,

$$\rho_y R(x) = \rho_y(L^*(x^*)^*) = (\lambda_y^* L^*(x^*))^* = (\rho_x^*(L(y)^*))^* = \lambda_x L(y),$$

and $(L, R) \in \mathfrak{M}_b(\mathbf{A})$.

• We can show the equivalence $1 \Leftrightarrow 3$ just as above. ■

With the same arguments as in the above Proposition, we can show an analogue statement in the unbounded case.

Proposition 3.6 *Let $L, R \in \mathcal{L}^+(\mathbf{A})$. We have the equivalence between*

1. (L, R) is a multiplier in $\mathfrak{M}(\mathbf{A})$.
2. L is in $\mathfrak{M}_L(\mathbf{A})$ and $R = JL^+J$.
3. R is in $\mathfrak{M}_R(\mathbf{A})$ and $L = JR^+J$.

3.2 Structure of bounded multipliers

In this section, we will show that $\mathfrak{M}_b(\mathbf{A})$ is a von Neumann algebra and that it is isomorphic to the left $\lambda(\mathbf{A})$ or the right $\rho(\mathbf{A})$ von Neumann algebra associated to the Hilbert algebra \mathbf{A} .

First we endow $\mathfrak{M}_b(\mathbf{A})$ with the following structure: if $T_i = (L_i, R_i) \in \mathfrak{M}_b(\mathbf{A})$ and $\mu \in \mathbb{C}$,

- vector space: $T_1 + \mu T_2 := (L_1 + \mu L_2, R_1 + \mu R_2)$,
- product: $T_1 T_2 := (L_1 L_2, R_2 R_1)$,
- adjoint: $T^* := (L^*, R^*)$,
- norm: $\|T\| := \|L\| = \|R\|$, since $R = JL^*J$ (see Proposition 3.5) and J conserves the norm.

Lemma 3.7 *For x in \mathbf{A}_b , we define $\Xi_x := (\lambda_x, \rho_x)$. Then $\Xi_x \in \mathfrak{M}_b(\mathbf{A})$. The map $\Xi : \mathbf{A}_b \rightarrow \mathfrak{M}_b(\mathbf{A})$ is an injective $*$ -algebra homomorphism. Moreover, for any $T = (L, R)$ in $\mathfrak{M}_b(\mathbf{A})$ and $x \in \mathbf{A}_b$, we have*

$$T\Xi_x = \Xi_{L(x)}, \quad \Xi_x T = \Xi_{R(x)}.$$

PROOF Let us show this Lemma in the bounded case. For $x \in \mathbf{A}_b$ and $y, z \in \mathbf{A}$, $\lambda_y \lambda_x(z) = \lambda_y \rho_z(x) = \rho_z \lambda_y(x) = \rho_z \rho_x(y)$. This means that $\Xi_x := (\lambda_x, \rho_x)$ lies in $\mathfrak{M}_b(\mathbf{A})$. Due to properties of λ and ρ , $\Xi : x \mapsto \Xi_x$ is a $*$ -algebra homomorphism. Moreover if $\Xi_x = 0$, it is immediate by Corollary 2.4 to see that $x = 0$, so that this homomorphism is injective. By using Lemma 3.3 as well as Definition 3.2 and Proposition 3.5, we find that $T\Xi_x = \Xi_{L(x)}$ and $\Xi_x T = \Xi_{R(x)}$. The arguments are similar in the unbounded case. ■

Lemma 3.8 *Endowed with the above structure, $\mathfrak{M}_b(\mathbf{A})$ is a von Neumann algebra.*

PROOF Indeed, $T \in \mathfrak{M}_b(\mathbf{A}) \mapsto \|T\|$ is a norm of algebra:

$$\|T_1 + T_2\| \leq \max(\|L_1\| + \|L_2\|, \|R_1\| + \|R_2\|) \leq \max(\|L_1\|, \|R_1\|) + \max(\|L_2\|, \|R_2\|) = \|T_1\| + \|T_2\|.$$

In the same way, $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$. Moreover, $(T_1 T_2)^* = T_2^* T_1^*$ and

$$\|T^* T\| = \max(\|L^* L\|, \|R R^*\|) = \max(\|L\|^2, \|R\|^2) = \max(\|L\|, \|R\|)^2 = \|T\|^2.$$

Let us now show that $\mathfrak{M}_b(\mathbf{A})$ is closed in $\mathcal{B}(\mathcal{H}_{\mathbf{A}}) \oplus \mathcal{B}(\mathcal{H}_{\mathbf{A}})$ for the weak topology. Consider a sequence $(T_n) = (L_n, R_n)$ of $\mathfrak{M}_b(\mathbf{A})$ that converges to $T = (L, R) \in \mathcal{B}(\mathcal{H}_{\mathbf{A}}) \oplus \mathcal{B}(\mathcal{H}_{\mathbf{A}})$, we want to show that $T \in \mathfrak{M}_b(\mathbf{A})$. Indeed,

$$\begin{aligned} |\langle z, \lambda_x L(y) - \rho_y R(x) \rangle| &\leq |\langle z, \lambda_x L(y) - \lambda_x L_n(y) \rangle| + |\langle z, \rho_y R_n(x) - \rho_y R(x) \rangle| \\ &\leq \|\lambda_x\| |\langle z, (L - L_n)(y) \rangle| + \|\rho_y\| |\langle z, (R_n - R)(x) \rangle| \rightarrow 0 \end{aligned}$$

when $n \rightarrow \infty$. So $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})$ and $\mathfrak{M}_b(\mathbf{A})$ is a von Neumann algebra, with unit $\mathbb{1} := (\text{id}_{\mathcal{H}_{\mathbf{A}}}, \text{id}_{\mathcal{H}_{\mathbf{A}}})$. ■

Theorem 3.9 *The bounded multiplier algebra of a Hilbert algebra \mathbf{A} satisfies*

$$\mathfrak{M}_b(\mathbf{A}) = \Xi(\mathbf{A}_b) := \{\Xi_x, x \in \mathbf{A}_b\}''.$$

PROOF Due to Lemma 3.7, $\Xi : \mathbf{A}_b \rightarrow \mathfrak{M}_b(\mathbf{A})$ is an injective algebra morphism and we see that $\{\Xi_x, x \in \mathbf{A}_b\} \simeq \mathbf{A}_b$ is a $*$ -ideal of the von Neumann algebra $\mathfrak{M}_b(\mathbf{A})$, by using also Lemma 3.8, so its weak closure $\Xi(\mathbf{A}) = \Xi(\mathbf{A}_b)$ is also a $*$ -ideal. Let (L_1, R_1) be the unit of $\Xi(\mathbf{A})$. Then for any $x, y \in \mathbf{A}$,

$$L_1(xy) = L_1\lambda_x(y) = \lambda_x(y) = xy, \quad R_1(yx) = R_1\rho_x(y) = \rho_x(y) = yx.$$

This means that L_1 and R_1 are the identity on the space $\{xy, x, y \in \mathbf{A}\}$ that is dense in \mathbf{A} by 4th axiom of Definition 2.1. Therefore, $L_1 = R_1 = \text{id}_{\mathcal{H}_{\mathbf{A}}}$. Finally, the unit $(\text{id}_{\mathcal{H}_{\mathbf{A}}}, \text{id}_{\mathcal{H}_{\mathbf{A}}})$ of $\mathfrak{M}_b(\mathbf{A})$ is contained in its $*$ -ideal $\Xi(\mathbf{A})$, so that $\mathfrak{M}_b(\mathbf{A}) = \Xi(\mathbf{A})$. ■

As a consequence, we have $\lambda(\mathbf{A}) = \mathfrak{M}_{L,b}(\mathbf{A})$ and $\rho(\mathbf{A}) = \mathfrak{M}_{R,b}(\mathbf{A})$. We see in particular with this characterization that the multiplier algebra of Hilbert-Schmidt operators is $\mathfrak{M}_b(\mathcal{L}^2(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H})$, due to the identification between bounded operators on \mathcal{H} and bounded operators on $\mathcal{L}^2(\mathcal{H})$.

Proposition 3.10 *We can describe the commutant of $\mathfrak{M}_b(\mathbf{A})$ by*

$$\mathfrak{M}_b(\mathbf{A})' = \left\{ \begin{pmatrix} R_1 & R_2J \\ JR_3 & JR_4J \end{pmatrix}, R_i \in \mathfrak{M}_{R,b}(\mathbf{A}) \right\},$$

as subspace of $\mathcal{B}(\mathcal{H}_{\mathbf{A}} \hat{\otimes} \mathcal{H}_{\mathbf{A}})$.

PROOF We represent a multiplier $(L, R) \in \mathfrak{M}_b(\mathbf{A})$ by the matrix $T = \begin{pmatrix} L & 0 \\ 0 & J LJ \end{pmatrix}$ in $\mathcal{B}(\mathcal{H}_{\mathbf{A}} \hat{\otimes} \mathcal{H}_{\mathbf{A}})$.

Let $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ in $\mathfrak{M}_b(\mathbf{A})'$ also seen as a subspace of $\mathcal{B}(\mathcal{H}_{\mathbf{A}} \hat{\otimes} \mathcal{H}_{\mathbf{A}})$. The commutation relation $TS = ST$ can be expressed as: for any $L \in \mathfrak{M}_{L,b}(\mathbf{A})$,

$$S_1L = LS_1, \quad S_2J LJ = LS_2, \quad S_3L = J LJ S_3, \quad S_4J LJ = J LJ S_4.$$

By using the usual commutant theorem, this means for example that S_1 lies in $\lambda(\mathbf{A})' = \rho(\mathbf{A})$. By Proposition 3.6, S_1 is then a left \mathbf{A} -module homomorphism, i.e. $S_1 \in \mathfrak{M}_{R,b}(\mathbf{A})$. We can use the same arguments for S_2J , JS_3 and JS_4J . ■

Moreover, we can consider the natural semifinite faithful normal trace on $\mathfrak{M}_b(\mathbf{A})^+$ given by: for any $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})^+$,

$$\tau(T) := \tau_\lambda(L) = \tau_\rho(R) = \langle x, x \rangle,$$

if $L^{\frac{1}{2}} = \lambda_x$ that is equivalent to $R^{\frac{1}{2}} = \rho_x$. Otherwise, it is given by $\tau(T) := +\infty$.

Due to the identification with left or right von Neumann algebras (see Theorem 3.9), we have now the following results for the bounded multipliers of a direct sum or a tensor product of Hilbert algebras.

Proposition 3.11 *Let \mathbf{A} and \mathbf{B} be two Hilbert algebras. Then we have $\mathfrak{M}_b(\mathbf{A} \oplus \mathbf{B}) \simeq \mathfrak{M}_b(\mathbf{A}) \oplus \mathfrak{M}_b(\mathbf{B})$.*

Proposition 3.12 *Let \mathbf{A} and \mathbf{B} be two Hilbert algebras. Then we have $\mathfrak{M}_b(\mathbf{A} \otimes \mathbf{B}) \simeq \mathfrak{M}_b(\mathbf{A}) \bar{\otimes} \mathfrak{M}_b(\mathbf{B})$, where $\bar{\otimes}$ denotes the tensor product of von Neumann algebras.*

3.3 Structure of unbounded multipliers

In this section, we will show that the space of (unbounded) multipliers $\mathfrak{M}(\mathbf{A})$ on a Hilbert algebra \mathbf{A} possesses a structure of pre-GW*-algebra strongly related to the von Neumann algebra of bounded multipliers $\mathfrak{M}_b(\mathbf{A})$. As in section 3.2, we endow $\mathfrak{M}(\mathbf{A})$ with basic operations: if $T_i = (L_i, R_i) \in \mathfrak{M}(\mathbf{A})$ and $\mu \in \mathbb{C}$,

- vector space: $T_1 + \mu T_2 := (L_1 + \mu L_2, R_1 + \mu R_2)$,
- product: $T_1 T_2 := (L_1 L_2, R_2 R_1)$,
- restriction of the adjoint to \mathbf{A} : $T^+ := (L^+, R^+)$.

Lemma 3.13 *For $x \in \mathbf{A}$, the restriction of Ξ_x to $\mathbf{A} \oplus \mathbf{A}$ lies in $\mathfrak{M}(\mathbf{A})$. The map $\Xi : \mathbf{A} \rightarrow \mathfrak{M}(\mathbf{A})$ is an injective *-algebra homomorphism. Moreover, for any $T = (L, R)$ in $\mathfrak{M}(\mathbf{A})$ and $x \in \mathbf{A}$, we have*

$$T \Xi_x = \Xi_{L(x)}, \quad \Xi_x T = \Xi_{R(x)}.$$

PROOF We can proceed as in the proof of Lemma 3.7. ■

Proposition 3.14 *For the above operations, $\mathfrak{M}(\mathbf{A})$ is an O*-algebra of domain $\mathbf{A} \oplus \mathbf{A}$, containing \mathbf{A} as a *-ideal.*

PROOF Let us show that $\mathfrak{M}(\mathbf{A})$ is a *-subalgebra of the O*-algebra $\mathcal{L}^+(\mathbf{A}) \oplus \mathcal{L}^+(\mathbf{A})$ containing the identity. We see easily that it is a vector subspace containing $(\mathbb{1}_{\mathbf{A}}, \mathbb{1}_{\mathbf{A}})$. Let $T_i = (L_i, R_i) \in \mathfrak{M}(\mathbf{A})$. Then $\forall x, y \in \mathbf{A}$, $x L_1 L_2(y) = R_1(x) L_2(y) = R_2 R_1(x) y$, so $T_1 T_2 \in \mathfrak{M}(\mathbf{A})$. Moreover if $T = (L, R) \in \mathfrak{M}(\mathbf{A})$, then $L^+ = J R J$ and $R^+ = J L J$ due to Proposition 3.6, so $T^+ = (L^+, R^+)$ satisfies the identity of multipliers, and $\mathfrak{M}(\mathbf{A})$ is an O*-algebra. Lemma 3.13 shows that $\mathbf{A} \simeq \{\Xi_x, x \in \mathbf{A}\}$ is a *-ideal of $\mathfrak{M}(\mathbf{A})$. ■

Contrary to the bounded multipliers, $\mathfrak{M}(\mathbf{A})$ depends strongly on \mathbf{A} and not only on \mathbf{A}_b .

Lemma 3.15 *The weak bounded commutant of $\mathfrak{M}(\mathbf{A})$ coincides with the commutant of $\mathfrak{M}_b(\mathbf{A})$: $\mathfrak{M}(\mathbf{A})'_w = \mathfrak{M}_b(\mathbf{A})'$.*

PROOF As in the proof of Proposition 3.10, we represent operators $T = (L, R) \in \mathfrak{M}(\mathbf{A})$ as matrices $T = \begin{pmatrix} L & 0 \\ 0 & R^+ \end{pmatrix}$ (with $R^+ = J L J$) in $\mathcal{L}^+(\mathbf{A} \oplus \mathbf{A})$. Let $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ be an element of $\mathfrak{M}(\mathbf{A})'_w$. The commutation relation of Definition 2.11, i.e. $\forall T = (L, R) \in \mathfrak{M}(\mathbf{A})$, $\forall x_i, y_i \in \mathbf{A}$

$$\langle (y_1, y_2), S T (x_1, x_2) \rangle = \langle T^+ (y_1, y_2), S (x_1, x_2) \rangle,$$

leads to the following system of equations

$$\begin{aligned} \langle y_1, S_1 L(x_1) \rangle &= \langle L^+(y_1), S_1(x_1) \rangle, & \langle y_2, S_3 L(x_1) \rangle &= \langle J L^+ J(y_2), S_3(x_1) \rangle, \\ \langle y_1, S_2 J L J(x_2) \rangle &= \langle L^+(y_1), S_2(x_2) \rangle, & \langle y_2, S_4 J L J(x_2) \rangle &= \langle J L^+ J(y_2), S_4(x_2) \rangle. \end{aligned}$$

In particular, for $L = \lambda_z$ with $z \in \mathbf{A}$, we have

$$\langle y, S_1(zx) \rangle = \langle y, S_1 L(x) \rangle = \langle L^+(y), S_1(x) \rangle = \langle z^* y, S_1(x) \rangle = \langle y, \lambda_z S_1(x) \rangle,$$

so that S_1 lies in $\mathfrak{M}_{R,b}(\mathbf{A})$. We can proceed in a similar way for $S_2 J$, $J S_3$ and $J S_4 J$.

Reciprocally, let $R_1 \in \mathfrak{M}_{R,b}(\mathbf{A})$, there exist $y_n \in \mathbf{A}$ such that $\rho_{y_n} \rightarrow R_1$ for the weak topology (see Theorem 3.9). Then $\forall x, z \in \mathbf{A}, \forall L \in \mathfrak{M}_L(\mathbf{A})$

$$\langle z, R_1 L(x) \rangle = \lim_{n \rightarrow \infty} \langle z, \rho_{y_n} L(x) \rangle = \lim_{n \rightarrow \infty} \langle z, L(x y_n) \rangle = \langle L^+(z), R_1(x) \rangle.$$

Therefore, $\begin{pmatrix} R_1 & R_2 J \\ J R_3 & J R_4 J \end{pmatrix}$, for $R_i \in \mathfrak{M}_{R,b}(\mathbf{A})$, are elements of the commutant $\mathfrak{M}(\mathbf{A})'_w$. \blacksquare

Lemma 3.16 *For any multiplier $T = (L, R) \in \mathfrak{M}(\mathbf{A})$, its closure, defined by $\bar{T} := (\bar{L}, \bar{R})$, is affiliated with the von Neumann algebra $\mathfrak{M}_b(\mathbf{A})$.*

PROOF First, we consider the closure \bar{L} of $L \in \mathfrak{M}_L(\mathbf{A})$. Let $x \in \text{Dom}(\bar{L})$, there exist $x_n \in \mathbf{A}$ such that $x_n \rightarrow x$ and $L(x_n) \rightarrow \bar{L}(x)$. Then for any $y \in \mathbf{A}$, we have $\rho_y(x_n) \rightarrow \rho_y(x)$ and $L(\rho_y(x_n)) = \rho_y L(x_n) \rightarrow \rho_y \bar{L}(x)$ due to Definition 3.4. Since \bar{L} is closed, $\rho_y(x) \in \text{Dom}(\bar{L})$ for $x \in \text{Dom}(\bar{L})$ and $y \in \mathbf{A}$, and $\bar{L}(\rho_y(x)) = \rho_y(\bar{L}(x))$.

Let $R_1 \in \mathfrak{M}_{R,b}(\mathbf{A})$, there exist $y_n \in \mathbf{A}$ such that $\rho_{y_n} \rightarrow R_1$ for the strong topology (see Theorem 3.9). Then for $x \in \text{Dom}(\bar{L})$, $\rho_{y_n}(x) \rightarrow R_1(x)$ and $\bar{L}(\rho_{y_n}(x)) = \rho_{y_n} \bar{L}(x) \rightarrow R_1(\bar{L}(x))$. So $\forall x \in \text{Dom}(\bar{L})$,

$$R_1(x) \in \text{Dom}(\bar{L}) \text{ and } \bar{L}(R_1(x)) = R_1 \bar{L}(x). \quad (3.1)$$

Now consider $T \in \mathfrak{M}(\mathbf{A})$ and its closure $\bar{T} = (\bar{L}, \bar{R})$. As $\bar{R}^\top = J \bar{L} J$, \bar{T} can be represented as $\begin{pmatrix} \bar{L} & 0 \\ 0 & J \bar{L} J \end{pmatrix}$ in the closed operators on $\mathcal{H}_{\mathbf{A}} \hat{\oplus} \mathcal{H}_{\mathbf{A}}$. By using the expression of the elements $\begin{pmatrix} R_1 & R_2 J \\ J R_3 & J R_4 J \end{pmatrix}$ of $\mathfrak{M}_b(\mathbf{A})'$ (see Proposition 3.10), we see that \bar{T} is affiliated with $\mathfrak{M}_b(\mathbf{A})$ (see Definition 2.12) if for any $R_i \in \mathfrak{M}_{R,b}(\mathbf{A})$,

$$R_1 \bar{L} \subset \bar{L} R_1, \quad R_2 \bar{L} J \subset \bar{L} R_2 J, \quad J R_3 \bar{L} \subset J \bar{L} R_3, \quad J R_4 \bar{L} J \subset J \bar{L} R_4 J.$$

And this is a consequence of Equation (3.1). \blacksquare

Lemma 3.17 *The unbounded bicommutant of $\mathfrak{M}(\mathbf{A})$ coincides with the multipliers: $\mathfrak{M}(\mathbf{A})''_{wc} = \mathfrak{M}(\mathbf{A})$.*

PROOF Let $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ in $\mathfrak{M}(\mathbf{A})''_{wc}$, so $S_i \in \mathcal{L}^+(\mathbf{A})$. Definition 2.14 of the bicommutant implies the following system of equations: for any $R_i \in \mathfrak{M}_{R,b}(\mathbf{A}), \forall x_i, y_i \in \mathbf{A}$

$$\begin{aligned} \langle R_1^*(y_1), S_1(x_1) \rangle + \langle J R_2^*(y_1), S_3(x_1) \rangle &= \langle S_1^+(y_1), R_1(x_1) \rangle + \langle S_2^+(y_1), J R_3(x_1) \rangle, \\ \langle R_1^*(y_1), S_2(x_2) \rangle + \langle J R_2^*(y_1), S_4(x_2) \rangle &= \langle S_1^+(y_1), R_2 J(x_2) \rangle + \langle S_2^+(y_1), J R_4 J(x_2) \rangle, \\ \langle R_3^* J(y_2), S_1(x_1) \rangle + \langle J R_4^* J(y_2), S_3(x_1) \rangle &= \langle S_3^+(y_2), R_1(x_1) \rangle + \langle S_4^+(y_2), J R_3(x_1) \rangle, \\ \langle R_3^* J(y_2), S_2(x_2) \rangle + \langle J R_4^* J(y_2), S_4(x_2) \rangle &= \langle S_3^+(y_2), R_2 J(x_2) \rangle + \langle S_4^+(y_2), J R_4 J(x_2) \rangle. \end{aligned}$$

since $\begin{pmatrix} R_1 & R_2 J \\ J R_3 & J R_4 J \end{pmatrix}$ are the elements of $\mathfrak{M}(\mathbf{A})'_w$ due to Lemma 3.15. By making vanish the appropriate R_i , we find the equations

$$\begin{aligned} \langle R_1^*(y), S_1(x) \rangle &= \langle S_1^+(y), R_1(x) \rangle, & \langle R_4^*(y), J S_4 J(x) \rangle &= \langle J S_4^+ J(y), R_4(x) \rangle, \\ \langle R_1^*(y), S_2(x) \rangle &= 0, & \langle J R_4^* J(y), S_3(x) \rangle &= 0, & \langle R_3^*(y), S_1(x) \rangle &= \langle J S_4^+ J(y), R_3(x) \rangle, \end{aligned}$$

for any $R_i \in \mathfrak{M}_{R,b}(\mathbf{A})$ and $\forall x, y \in \mathbf{A}$. By a similar argument as in the proof of Lemma 3.15, the first line of equations implies that S_1 and $J S_4 J$ lie in $\mathfrak{M}_L(\mathbf{A})$. Due to the fourth axiom of

Definition 2.1, the $R_1(y)$ generate a dense subspace of $\mathcal{H}_{\mathbf{A}}$ so we have $S_2 = S_3 = 0$. Finally, $\langle JS_4^+ J(y), R_3(x) \rangle = \langle R_3^*(y), JS_4 J(x) \rangle$ and $S_4 = JS_1 J$. We have thus showed that $S \in \mathfrak{M}(\mathbf{A})$. The reciproc $\mathfrak{M}(\mathbf{A}) \subset \mathfrak{M}(\mathbf{A})''_{wc}$ is evident. ■

Theorem 3.18 *Let \mathbf{A} be a Hilbert algebra. Then, $\mathfrak{M}(\mathbf{A})$ is a pre-GW*-algebra.*

PROOF In Lemmas 3.16 and 3.15, we have showed that $\forall T \in \mathfrak{M}(\mathbf{A})$, \bar{T} is affiliated with $\mathfrak{M}_b(\mathbf{A}) = (\mathfrak{M}(\mathbf{A})'_w)'$. Due to Proposition 2.13, we have $\mathfrak{M}(\mathbf{A})'_w(\mathbf{A} \oplus \mathbf{A}) \subset \widetilde{\mathbf{A} \oplus \mathbf{A}}$. Finally, by using Lemma 3.17, $\mathfrak{M}(\mathbf{A})''_{wc} = \mathfrak{M}(\mathbf{A})$ and the two conditions of Definition 2.16 are satisfied so that $\mathfrak{M}(\mathbf{A})$ is a pre-GW*-algebra. ■

As a consequence of this Theorem and of Proposition 2.17, from a Hilbert algebra \mathbf{A} we constructed a GW*-algebra given by the closure of $\mathfrak{M}(\mathbf{A})$. We also define the **bounded part of multipliers**: $\mathfrak{M}_{\text{stab}}(\mathbf{A}) := \mathfrak{M}_b(\mathbf{A})_{|(\mathbf{A} \oplus \mathbf{A})} \cap \mathfrak{M}(\mathbf{A})$. These are the bounded multipliers stabilizing \mathbf{A} as well as their adjoints. But in general, $\overline{\mathfrak{M}_{\text{stab}}(\mathbf{A})} \neq \mathfrak{M}_b(\mathbf{A})$.

3.4 Morphisms and unitaries

We show here the functorial property of the bounded and unbounded multipliers, and then we characterize unitary multipliers.

Definition 3.19 Let \mathbf{A} and \mathbf{B} be two Hilbert algebras. An **isomorphism** of Hilbert algebras between \mathbf{A} and \mathbf{B} is a *-algebra homomorphism $\Phi : \mathbf{A} \rightarrow \mathbf{B}$ that extends to a unitary map $\Phi : \mathcal{H}_{\mathbf{A}} \rightarrow \mathcal{H}_{\mathbf{B}}$. If $\mathbf{B} = \mathbf{A}$, we say that Φ is an automorphism of \mathbf{A} and we note $\Phi \in \text{Aut}(\mathbf{A})$. ♦

Proposition 3.20 *Let \mathbf{A} and \mathbf{B} be two Hilbert algebras. Given an isomorphism $\Phi : \mathbf{A} \rightarrow \mathbf{B}$ of Hilbert algebras, we define*

- the bounded extension $\tilde{\Phi} : \mathfrak{M}_b(\mathbf{A}) \rightarrow \mathfrak{M}_b(\mathbf{B})$ by

$$\forall T = (L, R) \in \mathfrak{M}_b(\mathbf{A}) \quad : \quad \tilde{\Phi}(T) := (\Phi \circ L \circ \Phi^{-1}, \Phi \circ R \circ \Phi^{-1}).$$

Then, $\tilde{\Phi}$ is a spatial isomorphism of von Neumann algebras.

- the unbounded extension $\tilde{\Phi} : \mathfrak{M}(\mathbf{A}) \rightarrow \mathfrak{M}(\mathbf{B})$ by

$$\forall T = (L, R) \in \mathfrak{M}(\mathbf{A}) \quad : \quad \tilde{\Phi}(T) := (\Phi \circ L \circ \Phi^{-1}, \Phi \circ R \circ \Phi^{-1}).$$

Then, $\tilde{\Phi}$ is a spatial isomorphism of pre-GW*-algebras.

PROOF Let us show this in the bounded case, the unbounded case is similar. First $\tilde{\Phi}(T)$ is a multiplier of \mathbf{B} . Indeed for any $x, y \in \mathbf{B}$,

$$\lambda_x \Phi L \Phi^{-1}(y) = \Phi(\lambda_{\Phi^{-1}(x)} L \Phi^{-1}(y)) = \Phi(\rho_{\Phi^{-1}(y)} R \Phi^{-1}(x)) = \rho_y \Phi R \Phi^{-1}(x).$$

It is straightforward to see that $\tilde{\Phi}$ is a morphism of algebras. It conserves the involutions since $\forall x, y \in \mathbf{B}$,

$$\langle \Phi L^* \Phi^{-1}(x), y \rangle = \langle L^* \Phi^{-1}(x), \Phi^{-1}(y) \rangle = \langle x, \Phi L \Phi^{-1}(y) \rangle. \quad \blacksquare$$

Note that if two Hilbert algebras \mathbf{A} and \mathbf{B} are isomorphic and Φ is the isomorphism, then the natural traces are compatible, i.e.

$$\forall T \in \mathfrak{M}_b(\mathbf{A}) \quad : \quad \tau_{\mathbf{B}}(\tilde{\Phi}(T)) = \tau_{\mathbf{A}}(T). \quad (3.2)$$

Proposition 3.21 *Let $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})$. Then, L is unitary if and only if R is unitary. Moreover, L stabilizes \mathbf{A} if and only if R stabilizes \mathbf{A} . We will therefore say that T is a **unitary multiplier** if $T \in \mathfrak{M}_{\text{stab}}(\mathbf{A})$ is unitary.*

PROOF Indeed for $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})$, $R = JL^*J$ and J is a unitary antilinear map. ■

Note that if $T \in \mathfrak{M}_{\text{stab}}(\mathbf{A})$ is a unitary multiplier and $\Phi : \mathbf{A} \rightarrow \mathbf{B}$ an isomorphism of Hilbert algebras, then $\tilde{\Phi}(T)$ is a unitary multiplier in $\mathfrak{M}_b(\mathbf{B})$ (see Proposition 3.20).

Proposition 3.22 *Let $T = (L, R) \in \mathfrak{M}_{\text{stab}}(\mathbf{A})$ be a unitary multiplier. Then, $U_T := LR^* = R^*L$ is an automorphism of the Hilbert algebra \mathbf{A} . Moreover, if T is involutive (i.e. $T^2 = \mathbb{1}$), so is U_T .*

PROOF Indeed, L and R^* are respectively in $\mathfrak{M}_{L,b}(\mathbf{A})$ and $\mathfrak{M}_{R,b}(\mathbf{A})$ and they commute together due to Theorem 3.9, and U_T is a unitary operator on $\mathcal{H}_{\mathbf{A}}$ stabilizing \mathbf{A} . Then, it preserves the involution

$$U_T(x)^* = (LR^*(x))^* = R^*L(x^*) = U_T(x^*),$$

for $x \in \mathbf{A}$, by using $L^*(x) = R(x^*)^*$. Moreover, $R^*(x)L(y) = RR^*(x)y = xy$ and

$$U_T(x)U_T(y) = (LR^*)(x)(LR^*)(y) = LR^*(R^*(x)L(y)) = U_T(xy). \quad \blacksquare$$

If an automorphism $U \in \text{Aut}(\mathbf{A})$ is associated to a unitary multiplier $T = (L, R) \in \mathfrak{M}_b(\mathbf{A})$ as above: $U = U_T$, we say that U is an **inner automorphism**. In this case, the induced spatial isomorphism of $\mathfrak{M}_b(\mathbf{A})$ has the form

$$\tilde{U}_T(T') = (LL'L^*, R^*R'R),$$

for any $T' = (L', R') \in \mathfrak{M}_b(\mathbf{A})$.

Proposition 3.23 *Let \mathbf{A} be a Hilbert algebra and $P = (P_L, P_R) \in \mathfrak{M}_b(\mathbf{A})$. Then P_L is a projection if and only if $P_R = JP_LJ$ is a projection. In this case, we note $\mathbf{B} := P_LP_R(\mathbf{A})$. \mathbf{B} is a Hilbert algebra dense in $\mathcal{H}_{\mathbf{B}} := P_LP_R(\mathcal{H}_{\mathbf{A}})$ and*

$$\mathfrak{M}_b(\mathbf{B}) = \{(P_LP_R L P_LP_R, P_LP_R R P_LP_R), \quad (L, R) \in \mathfrak{M}_b(\mathbf{A})\}.$$

PROOF Due to the formula $P_R = JP_L^*J$, we see that $P_L^2 = P_L = P_L^*$ if and only if $P_R^2 = P_R = P_R^*$. Then, the rest can be showed by using Lemma 3.7. ■

3.5 Topology on multipliers

The multiplier space $\mathfrak{M}(\mathbf{A})$ of an arbitrary Hilbert algebra \mathbf{A} is an O^* -algebra so that we can consider the graphic topology (see Definition 2.9) on its domain $\mathbf{A} \oplus \mathbf{A}$. However, we will define another more interesting topology on \mathbf{A} by introducing an auxiliary O^* -algebra called the derived multiplier algebra.

Definition 3.24 Let $\mathfrak{M}_{\text{topo}}(\mathbf{A})$ be the s^* -closure in $\mathcal{L}^+(\mathbf{A})$ of the vector space generated by LR' (for $L \in \mathfrak{M}_L(\mathbf{A})$ and $R' \in \mathfrak{M}_R(\mathbf{A})$), or equivalently the s^* -closure of the algebra generated by $\mathfrak{M}_L(\mathbf{A}) \oplus \mathfrak{M}_R(\mathbf{A})$. This is an O^* -algebra of domain \mathbf{A} called the **derived multiplier algebra**. Then, the **multiplier topology** $\tau_{\mathfrak{M}}$ on \mathbf{A} is defined as the graphic topology associated to the O^* -algebra $\mathfrak{M}_{\text{topo}}(\mathbf{A})$. \mathbf{A} is said to be **mult-closed** if it is complete for the multiplier topology. ♦

The seminorms of the multiplier topology are $\|x\|_T := \|T(x)\|$, for $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ and $x \in \mathbf{A}$.

Let us recall that $Z_{\mathbf{A}}$ denotes the bounded center (see Definition 2.7), and we define the **multiplier center** as $\mathcal{Z}_{\mathbf{b}}(\mathbf{A}) := \mathfrak{M}_{L,\mathbf{b}}(\mathbf{A}) \cap \mathfrak{M}_{R,\mathbf{b}}(\mathbf{A})$.

Lemma 3.25 *The bounded center $Z_{\mathbf{A}}$ is strongly dense in $\mathcal{Z}_{\mathbf{b}}(\mathbf{A})$.*

PROOF First, for any $z \in Z_{\mathbf{A}}$, $\lambda_z = \rho_z \in \mathcal{Z}_{\mathbf{b}}(\mathbf{A})$. Moreover $\forall T \in \mathcal{Z}_{\mathbf{b}}(\mathbf{A})$, $\lambda_{T(z)} = T\lambda_z = T\rho_z = \rho_{T(z)}$ by using Lemma 3.7. So $T(z) \in Z_{\mathbf{A}}$ and $Z_{\mathbf{A}}$ is a $*$ -ideal of $\mathcal{Z}_{\mathbf{b}}(\mathbf{A})$. Let T_1 denote the unit of the strong closure of $Z_{\mathbf{A}}$. As in Theorem 3.9, due to the density of $\{xy, x, y \in \mathbf{A}\}$ in $\mathcal{H}_{\mathbf{A}}$, we deduce that $T_1 = \text{id}_{\mathcal{H}_{\mathbf{A}}}$ and finally that $\mathcal{Z}_{\mathbf{b}}(\mathbf{A})$ is the strong closure of $Z_{\mathbf{A}}$. ■

Proposition 3.26 *Let \mathbf{A} be a Hilbert algebra. Then $\mathfrak{M}_{\text{topo}}(\mathbf{A})'_w = \mathcal{Z}_{\mathbf{b}}(\mathbf{A})$.*

PROOF Let $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})'_w$. Then, for any $x, y \in \mathbf{A}$ and $S \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$, we have $\langle y, TS(x) \rangle = \langle S^+(y), T(x) \rangle$. If we take $S = \lambda_z$ with $z \in \mathbf{A}$, we obtain that T commutes with λ_z , so $T \in \mathfrak{M}_{R,\mathbf{b}}(\mathbf{A})$. With $S = \rho_z$, we have $T \in \mathfrak{M}_{L,\mathbf{b}}(\mathbf{A})$, so $T \in \mathcal{Z}_{\mathbf{b}}(\mathbf{A}) \subset \mathfrak{M}_{\text{topo}}(\mathbf{A})'_w$. ■

Remark 3.27 Note that $\mathcal{Z}_{\mathbf{b}}(\mathbf{A})$ is a von Neumann algebra whose commutant is given by $\mathfrak{M}_{\text{topo},\mathbf{b}}(\mathbf{A})$, generated by LR' (for $L \in \mathfrak{M}_{L,\mathbf{b}}(\mathbf{A})$ and $R' \in \mathfrak{M}_{R,\mathbf{b}}(\mathbf{A})$). Due to the s^* -closure of $\mathfrak{M}_{\text{topo}}(\mathbf{A})$ and to Remark 2.15, we see that

$$\mathfrak{M}_{\text{topo}}(\mathbf{A}) = [\mathfrak{M}_{\text{topo},\mathbf{b}}(\mathbf{A})|_{\mathbf{A}}]^{s*} = [(\mathfrak{M}_{\text{topo}}(\mathbf{A})'_w)'|_{\mathbf{A}}]^{s*} = \mathfrak{M}_{\text{topo}}(\mathbf{A})''_{wc}. \quad \blacklozenge$$

We say that the Hilbert algebra \mathbf{A} is **centered** if $Z_{\mathbf{A}} \cdot \mathbf{A} \subset \mathbf{A}$.

Lemma 3.28 *Let \mathbf{A} be a centered Hilbert algebra. Then, for any $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$, the closure \overline{T} is affiliated with $\mathfrak{M}_{\text{topo},\mathbf{b}}(\mathbf{A})$.*

PROOF We proceed as for Lemma 3.16. Let $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ and $x \in \text{Dom}(\overline{T})$, there exists a sequence $x_n \in \mathbf{A}$ with $x_n \rightarrow x$ and $T(x_n) \rightarrow \overline{T}(x)$. For any $z \in Z_{\mathbf{A}}$, z is bounded and $\lambda_z(x_n) \rightarrow \lambda_z(x)$. Moreover, since $Z_{\mathbf{A}} \cdot \mathbf{A} \subset \mathbf{A}$, we deduce from Remark 3.27 that T commutes with λ_z and

$$T(\lambda_z(x_n)) = \lambda_z(T(x_n)) \rightarrow \lambda_z \overline{T}(x).$$

So $\lambda_z(x) \in \text{Dom}(\overline{T})$ and $\overline{T}(\lambda_z(x)) = \lambda_z \overline{T}(x)$. Due to Lemma 3.25, we can extend this result to all elements of $\mathcal{Z}_{\mathbf{b}}(\mathbf{A}) = \mathfrak{M}_{\text{topo},\mathbf{b}}(\mathbf{A})'$ instead of λ_z . ■

Proposition 3.29 *Let \mathbf{A} a centered mult-closed Hilbert algebra. Then, $\mathfrak{M}_{\text{topo}}(\mathbf{A})$ is a GW^* -algebra.*

PROOF Indeed, Lemma 3.28 with Proposition 2.13 show that $\mathfrak{M}_{\text{topo}}(\mathbf{A})'_w$ stabilizes \mathbf{A} . Finally, $\mathfrak{M}_{\text{topo}}(\mathbf{A})$ coincides with its bicommutant due to Remark 3.27. ■

Note that an element $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ decomposes as $T = \sum_i L_i R'_i$ (limit in the s^* -topology) with $L_i \in \mathfrak{M}_L(\mathbf{A})$ and $R'_i \in \mathfrak{M}_R(\mathbf{A})$, so that $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ if and only if $JTJ \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$.

Proposition 3.30 *Let us suppose that the multiplier topology is given by the seminorms generated by LR' , for a countable number of elements $L \in \mathfrak{M}(\mathbf{A})$, $R' \in \mathfrak{M}_R(\mathbf{A})$ (without s^* -limit). Then, $(\mathbf{A}, \tau_{\mathfrak{M}})$ is a topological $*$ -algebra, i.e. the product is separately continuous on $(\mathbf{A}, \tau_{\mathfrak{M}})$ and the involution is continuous on $(\mathbf{A}, \tau_{\mathfrak{M}})$. Moreover, if $(\mathbf{A}, \tau_{\mathfrak{M}})$ is complete, then it is a Fréchet algebra.*

PROOF Indeed for $x, y \in \mathbf{A}$ and $T = \sum_i L_i R'_i \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ (we can take the sum finite), we have $\|x^*\|_T = \|x\|_{JTJ}$ and

$$\|xy\|_T \leq \frac{1}{2}(\|\lambda_{L_i(x)}\| \|y\|_{R'_i} + \|\rho_{R'_i(y)}\| \|x\|_{L_i}). \quad \blacksquare$$

Note that the multiplier topology is automatically given by the topology associated to operators LR' (without s^* -limit) when this latter topology is Fréchet (see below Definition 2.9).

Proposition 3.31 *Let $\Phi : \mathbf{A} \rightarrow \mathbf{B}$ be an isomorphism of Hilbert algebras. Then, Φ is a homeomorphism for the multiplier topologies of \mathbf{A} and \mathbf{B} .*

PROOF We denote by $\tilde{\Phi} : \mathfrak{M}(\mathbf{A}) \rightarrow \mathfrak{M}(\mathbf{B})$ the unbounded extension of Φ (see Proposition 3.20). For $x \in \mathbf{A}$ and $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$, we have

$$\|x\|_T = \|T(x)\| = \|\Phi \circ T \circ \Phi^{-1}(\Phi(x))\| = \|\Phi(x)\|_{\tilde{\Phi}(T)},$$

and it turns out that $\Phi \circ T \circ \Phi^{-1} \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ due to the decomposition $T = \sum_i L_i R'_i$ (limit in the s^* -topology). \blacksquare

Proposition 3.32 *Let $(L, R) \in \mathfrak{M}(\mathbf{A})$. Then L and R are continuous linear maps $\mathbf{A} \rightarrow \mathbf{A}$ for the multiplier topology.*

PROOF Indeed for $T \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ and $x \in \mathbf{A}$, we have

$$\|L(x)\|_T = \|TL(x)\| = \|x\|_{TL}, \quad \|R(x)\|_T = \|TR(x)\| = \|x\|_{TR},$$

and as before, TL and TR are in $\mathfrak{M}_{\text{topo}}(\mathbf{A})$. \blacksquare

Definition 3.33 Let \mathbf{A} be a Hilbert algebra. We introduce here the **strong* topology** on the multipliers $\mathfrak{M}(\mathbf{A})$. Let $\mathfrak{B}_{\mathfrak{M}}$ be the space of bounded subsets of \mathbf{A} for the multiplier topology $\tau_{\mathfrak{M}}$. The seminorms of the strong* topology are

$$p_{B,S}(L) := \sup_{x \in B} \|SL(x)\| \quad \text{and} \quad p_{B,S}(R) = \sup_{x \in B} \|SR(x)\| = p_{JB,JSJ}(L^+),$$

for $(L, R) \in \mathfrak{M}(\mathbf{A})$, $B \in \mathfrak{B}_{\mathfrak{M}}$ and $S \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$. \blacklozenge

Proposition 3.34 *Endowed with the strong*-topology, $\mathfrak{M}(\mathbf{A})$ is a locally convex Hausdorff *-algebra. Moreover, if $(\mathbf{A}, \tau_{\mathfrak{M}})$ is Fréchet, then $\mathfrak{M}(\mathbf{A})$ is complete. If \mathbf{A} is Fréchet nuclear, $\mathfrak{M}(\mathbf{A})$ is complete nuclear.*

PROOF First, let us show that $\mathfrak{M}_L(\mathbf{A})$ is a closed subspace of $\mathcal{L}(\mathbf{A}, \tau_{\mathfrak{M}})$ that is a locally convex Hausdorff space. Let $L \in \mathcal{L}(\mathbf{A}, \tau_{\mathfrak{M}})$ and $L_n \in \mathfrak{M}_L(\mathbf{A})$ converging to L in the strong-topology. For any $x, y \in \mathbf{A}$, we choose a bounded subset $B \in \mathfrak{B}_{\mathfrak{M}}$ containing x and xy , we have

$$\|L(xy) - L(x)y\| \leq \|L(xy) - L_n(xy)\| + \|L_n(x)y - L(x)y\|$$

and $\|L_n(x)y - L(x)y\| \leq \|\rho_y\| \|L_n(x) - L(x)\|$, so that we obtain $L(xy) = L(x)y$ and $L \in \mathfrak{M}_L(\mathbf{A})$. We proceed in the same way for $\mathfrak{M}_R(\mathbf{A})$. And $\mathfrak{M}(\mathbf{A}) \simeq \mathfrak{M}_L(\mathbf{A}) \cap \mathfrak{M}_R(\mathbf{A})$ for the topologies described above. Moreover, the product is separately continuous on $\mathfrak{M}_L(\mathbf{A})$ and $\mathfrak{M}_R(\mathbf{A})$. Indeed, for $L, L' \in \mathfrak{M}_L(\mathbf{A})$,

$$p_{B,SL}(L') = p_{B,S}(LL') = p_{L'(B),S}(L),$$

with $SL \in \mathfrak{M}_{\text{topo}}(\mathbf{A})$ and $L'(B) \in \mathfrak{B}_{\mathfrak{M}}$.

If \mathbf{A} is Fréchet, then $\mathcal{L}(\mathbf{A}, \tau_{\mathfrak{M}})$ is complete (for the bounded convergence) and $\mathfrak{M}(\mathbf{A})$ also for the strong*-topology. If \mathbf{A} is Fréchet nuclear, then $\mathcal{L}(\mathbf{A}, \tau_{\mathfrak{M}})$ is complete nuclear and $\mathfrak{M}(\mathbf{A})$ also. \blacksquare

3.6 Special cases

Let us see first the case of a unital Hilbert algebra \mathbf{A} . From [4], we know that the left von Neumann associated to \mathbf{A} is finite, so $\mathfrak{M}_b(\mathbf{A}) \simeq \mathbf{A}_b$ is finite.

Proposition 3.35 *Let \mathbf{A} be a unital Hilbert algebra. Then $\mathfrak{M}(\mathbf{A}) \simeq \mathbf{A}$.*

PROOF Let indeed $T = (L, R) \in \mathfrak{M}(\mathbf{A})$. Then for any $x \in \mathbf{A}$, $L(x) = L(\mathbb{1})x \in \mathbf{A}$ and $L(\mathbb{1}) \in \mathbf{A}$. Moreover, $L(\mathbb{1}) = R(\mathbb{1})$. Therefore, $(L, R) \in \mathfrak{M}(\mathbf{A}) \mapsto L(\mathbb{1}) = R(\mathbb{1}) \in \mathbf{A}$ is an isomorphism. ■

We also see that $\mathfrak{M}(\mathbf{A})$ is not closed, so not a GW^* -algebra, unless $\mathbf{A} = \mathbf{A}_b = \mathcal{H}_{\mathbf{A}}$.

The second special case we consider here is commutative Hilbert algebras.

Proposition 3.36 *Let \mathbf{A} be a commutative Hilbert algebra. Then $\mathfrak{M}_b(\mathbf{A})$ is a commutative von Neumann algebra.*

PROOF First, for $x, y \in \mathbf{A}$ we have $\lambda_x(y) = \rho_x(y)$ and this extends by continuity to all $y \in \mathcal{H}_{\mathbf{A}}$. For $x \in \mathbf{A}_b$ and $y \in \mathbf{A}$, $\rho_x(y) = \lambda_y(x) = \rho_y(x) = \lambda_x(y)$ and this extends also to all $y \in \mathcal{H}_{\mathbf{A}}$. So $\mathbf{A}_b = Z_{\mathbf{A}}$ and $\mathfrak{M}_b(\mathbf{A}) = \mathcal{Z}_b(\mathbf{A})$ due to Theorem 3.9 and Lemma 3.25. ■

Proposition 3.37 *Let \mathbf{A} be a commutative Hilbert algebra. Then $\mathfrak{M}(\mathbf{A})$ is a commutative O^* -algebra and $\forall T \in \mathfrak{M}(\mathbf{A})$, $T^* = \overline{T}^+$.*

PROOF From Lemma 3.16, we have that $\overline{S}, \overline{T}$ are affiliated with $\mathfrak{M}_b(\mathbf{A})$ for any $S, T \in \mathfrak{M}(\mathbf{A})$. It means that they can be approximated in the s^* -topology by sequences S_n and T_m of $\mathfrak{M}_b(\mathbf{A})$. Since S_n and T_m commute by Proposition 3.36, we obtain by taking the limits that S and T commute.

The next part is standard and can be found in [9], but we indicate it here for self-containedness. For $T \in \mathfrak{M}(\mathbf{A})$ and $x \in \mathbf{A}$, we have $\|T(x)\| = \|T^+(x)\|$, so $\text{Dom}(\overline{T}) = \text{Dom}(\overline{T}^+)$. Let now $x \in \text{Dom}(T^*)$. We consider the polar decomposition $\overline{T} = U_T |\overline{T}|$ with U_T a unitary element of $\mathfrak{M}_b(\mathbf{A})$ (since \overline{T} is affiliated with). We have for any $y \in \mathbf{A}$,

$$\langle y, U_T T^* x \rangle = \langle \overline{T} U_T^* y, x \rangle = \langle U_T^* \overline{T} y, x \rangle = \langle |\overline{T}| y, x \rangle$$

since $U_T \in \mathfrak{M}_b(\mathbf{A}) \subset \mathfrak{M}_b(\mathbf{A})'$. Therefore, $x \in \text{Dom}(|\overline{T}|) = \text{Dom}(\overline{T}) = \text{Dom}(\overline{T}^+)$. ■

$\mathfrak{M}(\mathbf{A})$ is then called an essentially selfadjoint O^* -algebra.

Example 3.38 Let us consider the Schwartz function $\mathbf{A} = \mathcal{S}(\mathbb{R}^n)$ with commutative pointwise multiplication, complex conjugation and scalar product of $L^2(\mathbb{R}^n)$. \mathbf{A} is a commutative Hilbert algebra and its fulfillment is $\mathbf{A}_b = L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We have obviously $\mathfrak{M}_b(\mathbf{A}) \simeq L^\infty(\mathbb{R}^n)$ and $\mathfrak{M}(\mathbf{A}) \simeq \{T \in \mathcal{S}'(\mathbb{R}^n), \forall f \in \mathcal{S}(\mathbb{R}^{2n}), Tf = fT \in \mathcal{S}(\mathbb{R}^{2n})\}$: the tempered multipliers of \mathbf{A} . ♦

4 Multipliers in non-formal deformation quantization

4.1 Hilbert deformation quantization

Owing to the theory of multipliers and to the examples of non-formal deformation quantization we know (see next sections), we introduce the following definition of Hilbert deformation quantization.

Definition 4.1 Let M be a smooth manifold, and for any $\theta \in \mathbb{R}$, \mathbf{A}_θ be a subspace of complex measurable functions on M . Let \star_θ be an associative product on \mathbf{A}_θ for any $\theta \neq 0$ (\star_0 be the pointwise product on \mathbf{A}_0), and $\langle -, - \rangle$ be a fixed scalar product.

- The family $\mathbf{A} = (\mathbf{A}_\theta)$ is called a **Hilbert deformation quantization** (HDQ) of M if for any $\theta \in \mathbb{R}$, $(\mathbf{A}_\theta, \star_\theta)$ is a Hilbert algebra for the involution given by complex conjugation and for the scalar product $\langle -, - \rangle$, and if \mathbf{A}_θ contains the smooth functions with compact support $\mathcal{D}(M)$ as a dense subset.
- A Hilbert deformation quantization \mathbf{A} , with constant fibers $\mathbf{A}_\theta = \mathbf{A}$, is called **continuous** if for any $f_1, f_2 \in \mathbf{A}$, the map $\theta \mapsto \|f_1 \star_\theta f_2\|$ is continuous on \mathbb{R} , for the norm associated to the scalar product.
- Let \mathbf{A} and \mathbf{B} be two Hilbert deformation quantizations of the smooth manifolds M and N . An **intertwiner** of \mathbf{A} and \mathbf{B} is a family of isomorphisms $U_\theta : \mathbf{A}_\theta \rightarrow \mathbf{B}_\theta$ of Hilbert algebras (see Definition 3.19).
- A **representation** (also called a quantization map) of a HDQ \mathbf{A} is a family $\Omega = (\Omega_\theta)$ of isometric $*$ -morphisms $\Omega_\theta : \mathbf{A}_\theta \rightarrow \mathcal{L}^2(\mathcal{H}_\theta)$ with dense range, for (\mathcal{H}_θ) a family of Hilbert spaces. \blacklozenge

To any Hilbert deformation quantization (HDQ) $\mathbf{A} = (\mathbf{A}_\theta)$, we can associate directly the family of von Neumann algebras given by the bounded multipliers $\mathfrak{M}_b(\mathbf{A}) := (\mathfrak{M}_b(\mathbf{A}_\theta))$ and the family of pre-GW*-algebras given by the unbounded multipliers $\mathfrak{M}(\mathbf{A}) := (\mathfrak{M}(\mathbf{A}_\theta))$. Then, we can easily prove the following result.

Proposition 4.2 • If \mathbf{A} is a G -invariant HDQ, then $\mathfrak{M}_b(\mathbf{A})$ and $\mathfrak{M}(\mathbf{A})$ are also stabilized by the natural action of G .

- Let $U : \mathbf{A} \rightarrow \mathbf{B}$ be an intertwiner between two HDQ. Then, there exist $\tilde{T} : \mathfrak{M}_b(\mathbf{A}) \rightarrow \mathfrak{M}_b(\mathbf{B})$ a spatial isomorphism of von Neumann algebras and $\tilde{T} : \mathfrak{M}(\mathbf{A}) \rightarrow \mathfrak{M}(\mathbf{B})$ a spatial isomorphism of pre-GW*-algebras.

PROOF First, the action of G on the multipliers is $g^*T := g^* \circ T \circ (g^{-1})^*$, for $g \in G$ and $T \in \mathfrak{M}_b(\mathbf{A})$ or $T \in \mathfrak{M}(\mathbf{A})$. It turns out that g^* is an automorphism of any Hilbert algebra \mathbf{A}_θ due to the G -invariance, so we get the first result. The second result is just a translation of Proposition 3.20 in the framework of HDQ. \blacksquare

Proposition 4.3 Any representation Ω of a HDQ \mathbf{A} , $\Omega : \mathbf{A} \rightarrow \mathcal{L}^2(\mathcal{H})$, extends as an isomorphism of Hilbert algebras $\mathbf{A}_b \simeq \mathcal{L}^2(\mathcal{H})$ and then as a von Neumann isomorphism $\tilde{\Omega} : \mathfrak{M}_b(\mathbf{A}) \rightarrow \mathcal{B}(\mathcal{H})$.

PROOF This uses Proposition 4.2 and the standard fact that $\mathfrak{M}_b(\mathcal{L}^2(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H})$ for any Hilbert space \mathcal{H} . We recall the proof. Consider the map $\Phi : T \in \mathcal{B}(\mathcal{H}) \mapsto (T \circ \cdot, \cdot \circ T) \in \mathfrak{M}_b(\mathcal{L}^2(\mathcal{H}))$. It is an algebra homomorphism compatible with the involutions. But $\|T \circ \cdot\|_{\mathcal{B}(\mathcal{L}^2(\mathcal{H}))} = \|T\|_{\mathcal{B}(\mathcal{H})} = \|\cdot \circ T\|_{\mathcal{B}(\mathcal{L}^2(\mathcal{H}))}$ so that Φ is isometric.

Let (e_k) be a Hilbert basis of \mathcal{H} . Then, due to Parseval theorem, (φ_{kl}) defined by $\varphi_{kl}(e_m) := \delta_{lm}e_k$, is a Hilbert basis of $\mathcal{L}^2(\mathcal{H})$: any $S \in \mathcal{L}^2(\mathcal{H})$ decomposes as $S = \sum_{k,l} S_{kl}\varphi_{kl}$ with $\|S\|_2^2 = \sum_{k,l} |S_{kl}|^2 < \infty$. Any left multiplier $L \in \mathfrak{M}_{L,b}(\mathcal{L}^2(\mathcal{H}))$ writes $L(\varphi_{kl}) = \sum_{m,n} L_{mnkl}\varphi_{mn}$. Left multiplier condition as well as the identity $\varphi_{kl}\varphi_{mn} = \delta_{lm}\varphi_{kn}$ imply that $L_{mnkl} = L_{mk}\delta_{ln}$. By using the isometry of Φ , we deduce that L coincides with the image of Φ of the operator $e_k \mapsto \sum_m L_{mk}e_m$ in $\mathcal{B}(\mathcal{H})$, and Φ is surjective. \blacksquare

Proposition 4.4 • Let \mathbf{A} be a continuous HDQ. Then, the constant family \mathbf{A} is a lower semicontinuous family of pre-C*-algebras. But in general, its completion doesn't coincide with $\mathfrak{M}_b(\mathbf{A})$.

- Let $U : \mathbf{A} \rightarrow \mathbf{B}$ be an intertwiner between two HDQ and suppose that $\mathfrak{M}_b(\mathbf{A})$ is a continuous field of C^* -algebras. Then, $\mathfrak{M}_b(\mathbf{B})$ is also a continuous field of C^* -algebras.

PROOF The first result is due to M. Rieffel. If $\theta \rightarrow \|f_1 \star_\theta f_2\|$ is continuous for any $f_1, f_2 \in \mathbf{A}$, then $\langle f_1, f \star_\theta f_2 \rangle$ is also continuous and this implies lower semicontinuity (see [31]). The second result is obvious by using the isomorphism \tilde{U} . ■

4.2 Symmetries of Hilbert deformation quantizations

We assume here that the HDQ is complete, i.e. for any $\theta \in \mathbb{R}^*$, \mathbf{A}_θ is a complete Hilbert algebra. We introduce here the concept of symmetries of HDQ, which will be useful to construct other HDQ. Note that a symmetry of a HDQ \mathbf{A} of a manifold M does not come in general from a group action on M , but these group actions can be interesting examples.

Let $(\mathbf{A}_\theta, \star_\theta)$ be a complete HDQ of a smooth manifold M .

Definition 4.5 Let \mathfrak{g} be a countable-dimensional subspace of $\mathcal{C}^\infty(M)$, stable under complex conjugation, such that left and right \star_θ -multiplication L_T, R_T are defined as unbounded operators with domain containing $\mathcal{D}(M)$, for any $T \in \mathfrak{g}$. We assume that

1. \mathfrak{g} is a Lie algebra for the \star_θ -commutator, i.e. $\forall T_1, T_2 \in \mathfrak{g}, \exists T_3 \in \mathfrak{g}, [L_{T_1}, L_{T_2}] \subset L_{T_3}$ (covariance). Then, $\mathcal{U}(\mathfrak{g})$ has a countable PBW basis.
2. the \star_θ -multiplication L_T, R_T are unbounded operators with domain containing $\mathcal{D}(M)$, for any $T \in \mathcal{U}(\mathfrak{g})$,
3. such operators L_S, R_T commute on $\mathcal{D}(M)$ and act as multipliers on $\mathcal{D}(M)$,
4. they satisfy $JL_T J = R_{\bar{T}}$, where J is the complex conjugation,
5. and they commute with the bounded center $\mathcal{Z}_b(\mathbf{A})$.

Under these conditions, \mathfrak{g} is called a **symmetry** of the HDQ \mathbf{A} . ♦

Remark 4.6 If \mathfrak{g} contains functions such that L_T and R_T are in \mathbf{A}_θ , for any $T \in \mathcal{U}(\mathfrak{g})$, then conditions 2, 3, 4 and 5 are trivial. In the general unbounded case (for next theorem), we have to assume such conditions, but in concrete examples, with explicit expressions of L_T and R_T , these conditions will be easy to check. ♦

Theorem 4.7 Let \mathfrak{g} be a symmetry of the HDQ \mathbf{A} . Then, we set \mathbf{B}_θ to be the closure of $\mathcal{D}(M)$ for the seminorms $\|f\|_{S,T} := \|L_S R_T(f)\|$, for $S, T \in \mathcal{U}(\mathfrak{g})$ and $f \in \mathcal{D}(M)$.

Then, \mathbf{B}_θ is a dense Hilbert subalgebra of \mathbf{A}_θ and \mathbf{B} defines therefore a HDQ. Moreover, the above locally convex topology corresponds to the multiplier topology $\tau_{\mathfrak{M}}$ on \mathbf{B} , and $(\mathbf{B}, \tau_{\mathfrak{M}})$ is a Fréchet algebra.

PROOF First \mathbf{B}_θ is dense in \mathbf{A}_θ because it contains $\mathcal{D}(M)$. The algebra \mathbf{A}_θ is of type I_∞ so it is complete with respect to the scalar product, and hence the product \star_θ is jointly continuous for this topology. So for any $S, T \in \mathcal{U}(\mathfrak{g})$ and $f, g \in \mathcal{D}(M)$,

$$\|L_S R_T(f \star_\theta g)\| = \|L_S(f) \star_\theta R_T(g)\| \leq C \|L_S(f)\| \|R_T(g)\| < \infty,$$

due to condition 3 of Definition 4.5, and \mathbf{B}_θ is an algebra. Due to condition 4 of Definition 4.5, we have that $\|L_S R_T(\bar{f})\| = \|L_{\bar{T}} R_{\bar{S}}(f)\|$ and \mathbf{B}_θ is also stable by the complex conjugation.

We showed that \mathbf{B}_θ is a dense Hilbert subalgebra of \mathbf{A}_θ , but it has also a Fréchet topology with the seminorms $\|\cdot\|_{S,T}$. Moreover by definition, restricted to \mathbf{B}_θ , $L_T \in \mathfrak{M}_L(\mathbf{B}_\theta)$ and $R_T \in$

$\mathfrak{M}_R(\mathbf{B}_\theta)$. From condition 5 of Definition 4.5, we see by Propositions 3.29 and 3.26 that for any $S, T \in \mathcal{U}(\mathfrak{g})$, $L_S R_T \in \mathfrak{M}_{\text{topo}}(\mathbf{B}_\theta)$. It means that $\{(L_S)_{|\mathbf{B}_\theta}(R_T)_{|\mathbf{B}_\theta}, S, T \in \mathcal{U}(\mathfrak{g})\}$ generates an O^* -algebra on \mathbf{B}_θ contained in $\mathfrak{M}_{\text{topo}}(\mathbf{B}_\theta)$, and whose graphic topology is Fréchet. This topology then coincides with the multiplier topology of \mathbf{B}_θ by section 2.2. ■

In the above notations, we call \mathbf{B} the **Schwartz HDQ induced** by the symmetry \mathfrak{g} from the complete HDQ \mathbf{A} and denote it by $\mathbf{S}(\mathbf{A}, \mathfrak{g}) := \mathbf{B}$. Due to Proposition 3.34, $\mathfrak{M}(\mathbf{B})$ is then a family of locally convex complete $*$ -algebras, and it contains the symmetry \mathfrak{g} and its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Definition 4.8 Let $\Omega : \mathbf{B} \rightarrow \mathcal{L}^2(\mathcal{H})$ be a representation. We say that Ω is **extendable** if

- For any $T \in \mathfrak{M}(\mathbf{B})$, $\tilde{\Omega}_\theta(T)$ is defined as an unbounded operator on \mathcal{H}_θ with domain containing a common fixed dense subset \mathcal{D} .
- For any $T, S \in \mathfrak{M}(\mathbf{B})$, $\tilde{\Omega}_\theta(T)\tilde{\Omega}_\theta(S) = \tilde{\Omega}_\theta(T \star_\theta S)$.
- For any T , $\tilde{\Omega}_\theta(\bar{T}) = \tilde{\Omega}_\theta(T)^*$, where \bar{T} means the complex conjugate of T . ♦

If the associative product and the representation are given by smooth kernels as it is the case in various examples, these conditions are generally satisfied and easy to prove.

Theorem 4.9 Let \mathfrak{g} be a symmetry of the HDQ \mathbf{A} , $\mathbf{B} := \mathbf{S}(\mathbf{A}, \mathfrak{g})$ the Schwartz HDQ induced by \mathfrak{g} , and $\Omega : \mathbf{A} \rightarrow \mathcal{L}^2(\mathcal{H})$ a representation, which is extendable restricted to \mathbf{B} . Then, Ω extends to a faithful $*$ -representation $\tilde{\Omega} : \mathfrak{M}(\mathbf{B}) \rightarrow \mathcal{L}^+(\mathcal{D}_\mathbf{B})$, where $\mathcal{D}_\mathbf{B}$ is a dense domain canonically associated to \mathbf{B} .

PROOF First, we note that Ω is also a representation of the HDQ \mathbf{B} . Due to the proof of Theorem 4.7, $\mathfrak{M}(\mathcal{U}(\mathfrak{g}))$ is an O^* -algebra on \mathbf{B} . We then define the domain $\mathcal{D}_\mathbf{B}$ to be

$$\mathcal{D}_\mathbf{B} := \bigcap_{T \in \mathcal{U}(\mathfrak{g})} (\text{Dom}(\tilde{\Omega}(T)) \cap \text{Dom}(\tilde{\Omega}(T)^*)).$$

It is dense in \mathcal{H} as containing \mathcal{D} (see Definition 4.8). Due to the second condition of Definition 4.8 and to the fact that $\mathfrak{M}(\mathcal{U}(\mathfrak{g}))$ stabilizes \mathbf{B} by definition, we see that if $\varphi \in \mathcal{D}_\mathbf{B}$ and $f \in \mathbf{B}$, $\Omega(f)\varphi \in \mathcal{D}_\mathbf{B}$. So the maps $\Omega : \mathbf{B} \rightarrow \mathcal{L}^+(\mathcal{D}_\mathbf{B})$ and $\tilde{\Omega} : \mathfrak{M}(\mathcal{U}(\mathfrak{g})) \rightarrow \mathcal{L}^+(\mathcal{D}_\mathbf{B})$ are well-defined as a preliminary step.

Let us extend now these maps to the whole $\mathfrak{M}(\mathbf{B})$. For an element $\varphi \in \mathcal{D}_\mathbf{B}$, there exists $f_\varphi \in \mathbf{A}$ such that $\Omega(f_\varphi) = \frac{1}{\|\varphi\|^2} |\varphi\rangle\langle\varphi|$, since $\Omega : \mathbf{A} \rightarrow \mathcal{L}^2(\mathcal{H})$ is an isomorphism. Then, we have formally (if it is not infinite) for any $T = (L_T, R_T) \in \mathfrak{M}(\mathbf{B})$,

$$\begin{aligned} \|\tilde{\Omega}(T)\varphi\|_{\mathcal{H}}^2 &= \langle\varphi, \tilde{\Omega}(T)^+ \tilde{\Omega}(T)\varphi\rangle = \frac{1}{\|\varphi\|^2} \text{Tr}(|\varphi\rangle\langle\varphi|, \tilde{\Omega}(T)^+ \tilde{\Omega}(T)\varphi\langle\varphi|) = \|\tilde{\Omega}(T)\varphi\langle\varphi|\|_{\mathcal{L}^2(\mathcal{H})}^2 \\ &= \|\tilde{\Omega}(T)\tilde{\Omega}(f_\varphi)\|_{\mathcal{L}^2(\mathcal{H})}^2 = \|L_T(f_\varphi)\|_{\mathcal{H}_\mathbf{B}}^2, \\ \|\tilde{\Omega}(T)^+\varphi\|_{\mathcal{H}}^2 &= \langle\varphi, \tilde{\Omega}(T)\tilde{\Omega}(T)^+\varphi\rangle = \|R_T(f_\varphi)\|_{\mathcal{H}_\mathbf{B}}^2. \end{aligned}$$

A first application of this computation with $T \in \mathcal{U}(\mathfrak{g})$ shows that $L_T(f_\varphi)$ and $R_T(f_\varphi)$ are Hilbert-Schmidt, so $f_\varphi \in \mathbf{B}$. A second application with $T \in \mathfrak{M}(\mathbf{B})$ permits to show that $\mathcal{D}_\mathbf{B} \subset \text{Dom}(\tilde{\Omega}(T)) \cap \text{Dom}(\tilde{\Omega}(T)^*)$. Then, with a slight modification, we have

$$\begin{aligned} \|\tilde{\Omega}(T)\Omega(T')^*\Omega(S)\varphi\|_{\mathcal{H}}^2 &= \|L_T R_{T'} L_S(f_\varphi)\|_{\mathcal{H}_\mathbf{B}}^2 < \infty \\ \|\tilde{\Omega}(T)\Omega(T')^*\Omega(S)^*\varphi\|_{\mathcal{H}}^2 &= \|L_T R_{T'} R_S(f_\varphi)\|_{\mathcal{H}_\mathbf{B}}^2 < \infty \end{aligned}$$

for $T, T' \in \mathcal{U}(\mathfrak{g})$, $\varphi \in \mathcal{D}_{\mathbf{B}}$ and $S \in \mathfrak{M}(\mathbf{B})$, so $\tilde{\Omega}(S) \in \mathcal{L}^+(\mathcal{D}_{\mathbf{B}})$. Due to Definition 4.8, $\tilde{\Omega}$ is a $*$ -algebra homomorphism.

Let $S \in \mathfrak{M}(\mathbf{B})$ such that $\tilde{\Omega}(S) = 0$. Then, for any $f \in \mathbf{B}$, we have

$$\|L_S(f)\|_{\mathcal{H}_{\mathbf{B}}} = \|\Omega(L_S(f))\|_{\mathcal{L}^2(\mathcal{H})} = \|\tilde{\Omega}(S)\Omega(f)\| = 0,$$

due to the isometric map $\Omega : B \rightarrow \mathcal{L}^2(\mathcal{H})$. And $S = 0$, which shows the injectivity of $\tilde{\Omega}$. \blacksquare

4.3 Star-exponential

Let $(\mathbf{A}_{\theta}, \star_{\theta})$ be a complete HDQ of a smooth manifold M . Let also \mathfrak{g} be a real symmetry of \mathbf{A} and we note $\mathbf{B} := \mathcal{S}(\mathbf{A}, \mathfrak{g})$ the Schwartz HDQ induced by the symmetry \mathfrak{g} .

Definition 4.10 We define the **star-exponential** $E_{\star_{\theta}}(\frac{i}{\theta}T)$ of an element $T \in \mathfrak{g}$ to be the pair $(e^{\frac{i}{\theta}L_T}, e^{\frac{i}{\theta}R_T})$, where the exponential is understood in the sense of continuous functional calculus. \blacklozenge

Theorem 4.11 *The star-exponential of any element $T \in \mathfrak{g}$ is a unitary multiplier in $\mathfrak{M}_{\text{stab}}(\mathbf{B})$, and it satisfies the BCH property:*

$$\forall T, T' \in \mathfrak{g} \quad : \quad E_{\star_{\theta}}(\frac{i}{\theta}T) \star_{\theta} E_{\star_{\theta}}(\frac{i}{\theta}T') = E_{\star_{\theta}}(\frac{i}{\theta}BCH(T, T'))$$

where $BCH(T, T') = \log(e^T e^{T'})$ in the Lie algebra \mathfrak{g} .

PROOF Since any element $T \in \mathfrak{g}$ is a real smooth function, L_T, R_T are essentially selfadjoint operators so that $e^{\frac{i}{\theta}L_T}$ and $e^{\frac{i}{\theta}R_T}$ are unitary operators, and they are respectively equal to the strongly convergent series $\sum_{n=0}^{\infty} \frac{i^n}{n! \theta^n} (L_T)^n$ and $\sum_{n=0}^{\infty} \frac{i^n}{n! \theta^n} (R_T)^n$. The BCH property can be obtained by using this approximation by series as in formal deformation quantization (algebraic property).

What remains to be proved is the fact that $E_{\star_{\theta}}(\frac{i}{\theta}T)$ stabilizes \mathbf{B} . For example, for any $S, T \in \mathfrak{g}$ and $f \in \mathbf{B}$, we have

$$\|L_S e^{\frac{i}{\theta}L_T} f\| = \|e^{-\frac{i}{\theta}L_T} L_S e^{\frac{i}{\theta}L_T} f\| = \|L_{e^{-\frac{i}{\theta} \text{ad}(L_T)}(L_S)} f\| < \infty$$

since $e^{-\frac{i}{\theta} \text{ad}(L_T)}(L_S)$ is the left action of an element of \mathfrak{g} . In the same way, we can extend this result to L_S for $S \in \mathcal{U}(\mathfrak{g})$ and to R_S , which proves that $e^{\frac{i}{\theta}L_T} f \in \mathbf{B}$ by using Theorem 4.7. \blacksquare

Note that the star-exponential satisfies the following equation

$$\partial_t E_{\star_{\theta}}(\frac{it}{\theta}T) = \frac{i}{\theta}(L_T, R_T) E_{\star_{\theta}}(\frac{i}{\theta}T),$$

which could be useful to determine its explicit expression in the concrete examples.

4.4 Associated functional spaces

Let $(\mathbf{A}_{\theta}, \star_{\theta})$ be a complete HDQ of a smooth manifold M and let \mathfrak{g} be a real finite-dimensional symmetry of \mathbf{A} . We denote by $(e_i)_{i \in I}$ a basis of \mathfrak{g} (the following theory will be independent of its particular choice).

Definition 4.12 From this deformation \mathbf{A} and its symmetry \mathfrak{g} , we define

- the **Sobolev spaces** $H^k(\mathbf{A}, \mathfrak{g})$ ($k \in \mathbb{N}$) **induced** by \mathfrak{g} as the completion of $\mathcal{D}(M)$ for the norm

$$\|f\|_k := \sup_{l \leq k, i_1, \dots, i_l \in I} \|(L_{e_{i_1}} - R_{e_{i_1}})(L_{e_{i_2}} - R_{e_{i_2}}) \dots (L_{e_{i_l}} - R_{e_{i_l}})f\|.$$

- the **GBV spaces**² $G^{k,l}(\mathbf{A}, \mathfrak{g})$ ($k, l \in \mathbb{N}$) **induced** by \mathfrak{g} as the completion of $\mathcal{D}(M)$ for the norm

$$\|f\|_{k,l} := \sup_{p \leq k, i_1, \dots, i_p \in I} \sup_{q \leq l, j_1, \dots, j_q \in I} \|L_{e_{i_1}} \dots L_{e_{i_p}} R_{e_{j_1}} \dots R_{e_{j_q}} f\|. \quad \blacklozenge$$

To give an intuition, note that $(L_{e_i} - R_{e_i})$ just corresponds to the inner \star_θ -derivation $[e_i, \cdot]_{\star_\theta}$.

Proposition 4.13 *For $k \in \mathbb{N}$, the induced Sobolev space $H^k(\mathbf{A}_\theta, \mathfrak{g})$ is a dense Hilbert subalgebra of \mathbf{A}_θ and it is a Hilbert space for the scalar product*

$$\langle f_1, f_2 \rangle_k := \sum_{l=0}^k \sum_{i_1, \dots, i_l \in I} \langle (L_{e_{i_1}} - R_{e_{i_1}}) \dots (L_{e_{i_l}} - R_{e_{i_l}}) f_1, (L_{e_{i_1}} - R_{e_{i_1}}) \dots (L_{e_{i_l}} - R_{e_{i_l}}) f_2 \rangle.$$

PROOF For any $f_1, f_2 \in H^k(\mathbf{A}, \mathfrak{g})$, we can show recursively by using the Leibniz rule for the \star_θ -derivation $(L_{e_i} - R_{e_i})$ that $f_1 \star_\theta f_2 \in H^k(\mathbf{A}, \mathfrak{g})$. Indeed, at the first orders,

$$\begin{aligned} \|(L_{e_i} - R_{e_i})(f_1 \star_\theta f_2)\| &\leq \|(L_{e_i} - R_{e_i})f_1\| \|f_2\| + \|f_1\| \|(L_{e_i} - R_{e_i})f_2\| < +\infty, \\ \|(L_{e_{i_1}} - R_{e_{i_1}})(f_1 \star_\theta f_2)\| &\leq \|(L_{e_{i_1}} - R_{e_{i_1}})(L_{e_{i_2}} - R_{e_{i_2}})f_1\| \|f_2\| + \|(L_{e_{i_1}} - R_{e_{i_1}})f_1\| \|(L_{e_{i_2}} - R_{e_{i_2}})f_2\| \\ &\quad + \|(L_{e_{i_2}} - R_{e_{i_2}})f_1\| \|(L_{e_{i_1}} - R_{e_{i_1}})f_2\| + \|f_1\| \|(L_{e_{i_1}} - R_{e_{i_1}})(L_{e_{i_2}} - R_{e_{i_2}})f_2\|, \end{aligned}$$

where we also used that the Hilbert algebra \mathbf{A}_θ is complete, so the product \star_θ is jointly continuous for its Hilbert topology. Iteration of such identities yields the result: $f_1 \star_\theta f_2 \in H^k(\mathbf{A}, \mathfrak{g})$. For the involution J (corresponding to the complex conjugation), we have

$$\begin{aligned} \|(L_{e_{i_1}} - R_{e_{i_1}}) \dots (L_{e_{i_l}} - R_{e_{i_l}}) \bar{f}\| &= \|J(L_{e_{i_1}} - R_{e_{i_1}})JJ \dots JJ(L_{e_{i_l}} - R_{e_{i_l}})Jf\| \\ &= \|(L_{e_{i_1}} - R_{e_{i_1}}) \dots (L_{e_{i_l}} - R_{e_{i_l}})f\|, \end{aligned}$$

because J is unitary and involutive, and $JL_{e_i}J = R_{\bar{e}_i} = R_{e_i}$. So if $f \in H^k(\mathbf{A}, \mathfrak{g})$, its complex conjugate also belongs to the Sobolev space. It is then easy to see that $\langle -, - \rangle_k$ is a hermitian positive definite scalar product and that it is associated to the topology of $H^k(\mathbf{A}, \mathfrak{g})$. \blacksquare

Remark 4.14 We note $H^\infty(\mathbf{A}, \mathfrak{g}) = \bigcap_{k=0}^\infty H^k(\mathbf{A}, \mathfrak{g})$. It is also a dense Hilbert subalgebra of \mathbf{A} , and a Fréchet algebra for the projective limit topology. \blacklozenge

Proposition 4.15 *For $k, l \in \mathbb{N}$, the GBV induced spaces $G^{k,l}(\mathbf{A}, \mathfrak{g})$ satisfy*

$$G^{k,p}(\mathbf{A}_\theta, \mathfrak{g}) \star_\theta G^{q,l}(\mathbf{A}_\theta, \mathfrak{g}) \subset G^{k,l}(\mathbf{A}_\theta, \mathfrak{g}), \quad J(G^{k,l}(\mathbf{A}_\theta, \mathfrak{g})) = G^{l,k}(\mathbf{A}_\theta, \mathfrak{g}),$$

for any $p, q \in \mathbb{N}$, and where J is the complex conjugation. As a consequence, $G^{k,l}(\mathbf{A}_\theta, \mathfrak{g})$ is a subalgebra of \mathbf{A}_θ and a Hilbert space for the scalar product

$$\langle f_1, f_2 \rangle_{k,l} := \sum_{p=0}^k \sum_{q=0}^l \sum_{i_1, \dots, i_p \in I} \sum_{j_1, \dots, j_q \in I} \langle L_{e_{i_1}} \dots L_{e_{i_p}} R_{e_{j_1}} \dots R_{e_{j_q}} f_1, L_{e_{i_1}} \dots L_{e_{i_p}} R_{e_{j_1}} \dots R_{e_{j_q}} f_2 \rangle.$$

Moreover, $G^{k,k}(\mathbf{A}_\theta, \mathfrak{g})$ is a Hilbert subalgebra of \mathbf{A}_θ contained in $H^k(\mathbf{A}_\theta, \mathfrak{g})$.

²GBV stands for Gracia-Bondia-Varilly as we will see that these spaces generalize the spaces defined by GBV for the Moyal deformation by using the matrix basis.

PROOF By using the unbounded multiplier property of the symmetry: $L_{e_i} R_{e_j}(f_1 \star_\theta f_2) = L_{e_i}(f_1) \star_\theta R_{e_j}(f_2)$, as well as the completeness of the HDQ \mathbf{A}_θ , we obtain the result concerning the star-product of GBV spaces. Since the complex conjugation transforms a left multiplication into a right one, we have also that $J(G^{k,l}(\mathbf{A}_\theta, \mathfrak{g})) = G^{l,k}(\mathbf{A}_\theta, \mathfrak{g})$. The rest are easy consequences. ■

Remark 4.16 It turns out that the GBV spaces form a decreasing sequence, i.e. $G^{k',l'}(\mathbf{A}, \mathfrak{g}) \subset G^{k,l}(\mathbf{A}, \mathfrak{g})$ for $k \leq k'$ and $l \leq l'$. Moreover, we can see that $\mathcal{S}(\mathbf{A}, \mathfrak{g}) = \bigcap_{k,l \in \mathbb{N}} G^{k,l}(\mathbf{A}, \mathfrak{g})$ and that the projective limit topology corresponds to the standard topology defined in Theorem 4.7. ♦

We see in particular that the spaces $H^k(\mathbf{A}, \mathfrak{g})$ and $G^{k,k}(\mathbf{A}, \mathfrak{g})$ define HDQs, and also non-trivial unbounded multipliers \star -algebras.

Lemma 4.17 For any $k, l, p, q \in \mathbb{N}$, we have

$$G^{k,l}(\mathbf{A}, \mathfrak{g}) \star_\theta \mathfrak{M}_b(\mathcal{S}(\mathbf{A}, \mathfrak{g})) \star_\theta G^{p,q}(\mathbf{A}, \mathfrak{g}) \subset G^{k,q}(\mathbf{A}, \mathfrak{g}).$$

PROOF Let $f_1 \in G^{k,l}(\mathbf{A}, \mathfrak{g})$, $f_2 \in G^{p,q}(\mathbf{A}, \mathfrak{g})$ and $T \in \mathfrak{M}_b(\mathcal{S}(\mathbf{A}, \mathfrak{g}))$. Then, we have

$$\begin{aligned} \|L_{e_{i_1}} \dots L_{e_{i_p}} R_{e_{j_1}} \dots R_{e_{j_q}}(R_T(f_1) \star_\theta f_2)\| &\leq \|L_{e_{i_1}} \dots L_{e_{i_p}} R_T(f_1)\| \|R_{e_{j_1}} \dots R_{e_{j_q}} f_2\| \\ &\leq \|R_T\| \|L_{e_{i_1}} \dots L_{e_{i_p}} f_1\| \|R_{e_{j_1}} \dots R_{e_{j_q}} f_2\| < \infty, \end{aligned}$$

by using the fact that \mathbf{A}_θ is complete, and that $(R_T, L_T) \in \mathfrak{M}(\mathcal{S}(\mathbf{A}, \mathfrak{g}))'_w$ by Lemma 3.15 and Proposition 3.10, so that $L_{e_{i_1}} \dots L_{e_{i_p}}$ and R_T commute. ■

Proposition 4.18 The induced Schwartz space $\mathcal{S}(\mathbf{A}, \mathfrak{g})$ is stable under the holomorphic functional calculus.

PROOF We adapt the argument of [28] to this more general case. Let $f \in \mathcal{S}(\mathbf{A}, \mathfrak{g})$ such that $1 + f$ is invertible in $\mathfrak{M}_b(\mathbf{A})$, and we note $1 + T$ its inverse. Let us show that $T \in \mathcal{S}(\mathbf{A}, \mathfrak{g})$. Invertibility condition means $f + T + f \star_\theta T = 0$ in $\mathfrak{M}_b(\mathcal{S}(\mathbf{A}, \mathfrak{g}))$ where we identify f with its multiplier $\Xi_f = (\lambda_f, \rho_f)$. We multiply on the right by f and obtain

$$T = -f - T \star_\theta f = -f + f \star_\theta f + f \star_\theta T \star_\theta f.$$

By using Lemma 4.17 and Remark 4.16, we have that $f \star_\theta T \star_\theta f \in \bigcap_{k,l \in \mathbb{N}} G^{k,l}(\mathbf{A}, \mathfrak{g}) = \mathcal{S}(\mathbf{A}, \mathfrak{g})$; so $T \in \mathcal{S}(\mathbf{A}, \mathfrak{g})$, and $\mathcal{S}(\mathbf{A}, \mathfrak{g})$ is stable under holomorphic calculus. ■

4.5 Invariant symmetries

To motivate the definition of an invariant symmetry of a HDQ \mathbf{A} , let us consider first a formal deformation quantization \star_θ of a symplectic manifold (M, ω) , on which the Lie group G is acting in a strong Hamiltonian way, i.e. there exists moment maps $\eta_X \in \mathcal{C}^\infty(M)$ for any $X \in \mathfrak{g}$ (\mathfrak{g} : Lie algebra of G), such that the Poisson bracket with respect to ω gives $\{\eta_X, \eta_Y\} = \eta_{[X,Y]}$.

We recall that \star_θ is said to be covariant if $[\eta_X, \eta_Y]_{\star_\theta} = -i\theta\eta_{[X,Y]}$, and invariant if $\forall g \in G$ the pullback of the left action leaves the product invariant: $g^*(f_1 \star_\theta f_2) = (g^*f_1) \star_\theta (g^*f_2)$. The invariance at the infinitesimal level provides $X^*(f_1 \star_\theta f_2) = (X^*f_1) \star_\theta f_2 + f_1 \star_\theta (X^*f_2)$, for $X \in \mathfrak{g}$ and $X^* = \frac{d}{dt}|_{t=0}(e^{-tX})^*$ is the fundamental vector field.

Suppose that M is cohomologically trivial, so that any derivation is inner, there exist $\Xi_X \in \mathcal{C}^\infty(M)$ such that $X^* = \frac{i}{\theta} [\Xi_X, \cdot]_{\star_\theta}$. But we have

$$\frac{i}{\theta} [\Xi_X - \eta_X, \eta_Y]_{\star_\theta} = X^*(\eta_Y) - \eta_{[X,Y]} = 0.$$

If the Lie group G contains “sufficiently symmetries”, then $\Xi_X = \eta_X$, and $X^* = \frac{i}{\theta} [\eta_X, \cdot]_{\star_\theta}$, which will be a very useful property.

Reciprocally, if $X^* = \frac{i}{\theta} [\eta_X, \cdot]_{\star_\theta}$, then the star-product is obviously covariant and each fundamental vector is a \star_θ -derivation. So, if the Lie group G is connected, the pullback of the left action of G acts by \star_θ -automorphisms, so \star_θ is also G -invariant. This discussion comes from ideas of the retract method of P. Bieliavsky.

We can now introduce the notion of invariant symmetry. Let us consider now non-formal deformation quantization.

Definition 4.19 Let $(\mathbf{A}_\theta, \star_\theta)$ be a complete HDQ of a smooth manifold M . Let G be a connected Lie group acting smoothly and in a strong Hamiltonian way on M , with Lie algebra \mathfrak{g} . This action is said to be an **invariant symmetry** of the HDQ \mathbf{A}_θ if the left action \mathfrak{L}_g of $g \in G$, defined by $\mathfrak{L}_g(f)(x) := f(g^{-1}x)$ ($x \in M$), is an automorphism of the Hilbert algebra \mathbf{A}_θ and if

$$\forall X \in \mathfrak{g} \quad : \quad X^* = \frac{i}{\theta} [\eta_X, \cdot]_{\star_\theta},$$

where η_X denotes the moment map of $X \in \mathfrak{g}$. ◆

Given such a G -invariant symmetry of the HDQ \mathbf{A} , the moment maps η_X then form a symmetry of the HDQ \mathbf{A}_θ . We can consider the star-exponential of the moment maps $E_{\star_\theta}(\frac{i}{\theta}\eta_X)$ (see section 4.3) or its integrated version:

$$\mathcal{E}_{e^X} := E_{\star_\theta}(\frac{i}{\theta}\eta_X).$$

Proposition 4.20 Let \mathbf{A} be a complete HDQ with a G -invariant symmetry. Then the star-exponential induces an injective homomorphism $\mathcal{E} : G \rightarrow \mathfrak{M}_b(\mathbf{A})$, valued in the unitary multipliers stabilizing $\mathcal{S}(\mathbf{A}, \mathfrak{g})$, and satisfying

$$\forall f \in \mathbf{A}_\theta, \quad \forall g \in G \quad : \quad \mathfrak{L}_g(f) = \mathcal{E}_g \star_\theta f \star_\theta \mathcal{E}_{g^{-1}}. \quad (4.1)$$

PROOF The first part of the Proposition is an easy consequence of Theorem 4.11. Then, the infinitesimal version of Equation (4.1) is given by $X^* = \frac{i}{\theta} [\eta_X, \cdot]_{\star_\theta}$ since X^* is the derivative of $\mathfrak{L}_{e^{tX}}$ and η_X the one of $\mathcal{E}_{e^{tX}} = E_{\star_\theta}(\frac{i}{\theta}t\eta_X)$. Integration of this condition provides the result. ■

We say that an intertwiner $V : \mathbf{A} \rightarrow \mathbf{B}$, between two complete HDQ \mathbf{A} and \mathbf{B} with the same invariant G -symmetry, is **G -equivariant** if the map V_θ is equivariant for the actions of G on \mathbf{A}_θ and \mathbf{B}_θ .

Proposition 4.21 If the HDQs \mathbf{A} and \mathbf{B} have the same invariant G -symmetry and if $V : \mathbf{A} \rightarrow \mathbf{B}$ is a G -equivariant intertwiner, the bounded and unbounded extensions \tilde{V} are also G -equivariant.

PROOF Indeed, we compute that

$$\tilde{V}(\mathfrak{L}_g(T)) = V \circ \mathfrak{L}_g \circ T \circ \mathfrak{L}_{g^{-1}} \circ V^{-1} = \mathfrak{L}_g \circ V \circ T \circ V^{-1} \circ \mathfrak{L}_{g^{-1}} = \mathfrak{L}_g(\tilde{V}(T)),$$

for $g \in G$, $T \in \mathfrak{M}_b(\mathbf{A})$ or $\mathfrak{M}(\mathbf{A})$, due to G -invariant symmetry and to the equivariance of V . ■

Proposition 4.22 Let \mathbf{A} be a complete HDQ with a G -invariant symmetry and $\Omega : \mathbf{A} \rightarrow \mathcal{L}^2(\mathcal{H})$ a representation of \mathbf{A} . Then, $U_\theta(g) := \Omega_\theta(\mathcal{E}_g)$ defines a unitary representation $U_\theta : G \rightarrow \mathcal{U}(\mathcal{H}_\theta)$ and Ω is G -equivariant:

$$\forall f \in \mathbf{A}_\theta, \quad \forall g \in G \quad : \quad \Omega_\theta(\mathfrak{L}_g(f)) = U_\theta(g)\Omega_\theta(f)U_\theta(g^{-1}).$$

PROOF This is just a consequence of the definition of a representation (see Definition 4.1) and of Equation (4.1). \blacksquare

Proposition 4.23 *Let \mathbf{A} be a complete HDQ with a G -invariant symmetry. Then,*

- $\mathcal{S}(\mathbf{A}_\theta, \mathfrak{g})$ is the space of smooth vectors of \mathbf{A}_θ for the action $G \times G \rightarrow \text{Aut}(\mathbf{A}_\theta)$, $(g, g') \mapsto L_{\mathcal{E}_g} R_{\mathcal{E}_{g'}}^*$.
- $\mathcal{H}^k(\mathbf{A}_\theta, \mathfrak{g})$ is the space of \mathcal{C}^k -vectors of \mathbf{A}_θ for the action $G \rightarrow \text{Aut}(\mathbf{A}_\theta)$, $g \mapsto L_{\mathcal{E}_g} R_{\mathcal{E}_g}^*$. It is independent of the parameter θ .

PROOF This result is a consequence of the definition of

$$L_{\eta_X} = \frac{d}{dt}\bigg|_{t=0} L_{(\mathcal{E}_{e^{tX}})}, \quad R_{\eta_X} = \frac{d}{dt}\bigg|_{t=0} R_{(\mathcal{E}_{e^{tX}})}, \quad L_{\eta_X} - R_{\eta_X} = [\eta_X, \cdot]_{\star_\theta} = -i\theta X^*. \quad \blacksquare$$

5 Examples of Hilbert deformation quantization

5.1 Moyal-Weyl deformation quantization

We recall the non-formal expression of the Moyal product: $\forall f_1, f_2 \in \mathcal{D}(\mathbb{R}^{2n})$,

$$(f_1 \star_\theta f_2)(x) := \frac{1}{(\pi\theta)^{2n}} \int f_1(y) f_2(z) e^{-\frac{2i}{\theta}(\omega(y,z) + \omega(z,x) + \omega(x,y))} dy dz, \quad (5.1)$$

where ω is the standard symplectic form of \mathbb{R}^{2n} , $\theta \in \mathbb{R}^*$ and $x \in \mathbb{R}^{2n}$. It satisfies the tracial identity

$$\int (f_1 \star_\theta f_2)(x) dx = \int f_1(x) f_2(x) dx. \quad (5.2)$$

It is then well-known that the star-product \star_θ extends to $L^2(\mathbb{R}^{2n})$, and that space endowed with \star_θ , the complex conjugation and the standard scalar product $\langle f_1, f_2 \rangle := \int_{\mathbb{R}^{2n}} \overline{f_1(x)} f_2(x) dx$, is a full (and complete) Hilbert algebra. We then introduce its bounded multiplier algebra $\mathfrak{M}_b(L^2(\mathbb{R}^{2n}))$, and call it the **bounded Moyal multiplier algebra**. It is actually isomorphic to the left bounded multipliers (also used in [28]) and it is a von Neumann algebra of type I_∞ with trivial center $\mathcal{Z}_b(L^2(\mathbb{R}^{2n})) = \mathbb{C}1$.

We also recall the form of the Weyl quantization. If $x = (q, p) \in \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^{2n}$ that is a Lagrangian decomposition with respect to ω , the Weyl map $\Omega_\theta : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}^2(L^2(\mathbb{R}^n))$ is given by

$$\Omega_\theta(f)\varphi(q_0) := \frac{2^n}{(\pi\theta)^n} \int f(q, p) e^{\frac{4i}{\theta}\omega(q-q_0, p)} \varphi(2q - q_0) dq dp, \quad (5.3)$$

for $\varphi \in L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^{2n})$. See also [32] for study on the Weyl map.

Set $\mathbf{A}_\theta := (L^2(\mathbb{R}^{2n}), \star_\theta)$ for $\theta \neq 0$ and $\mathbf{A}_0 := L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ in this section.

Proposition 5.1 *The family $\mathbf{A} := (\mathbf{A}_\theta)$ defines a complete Hilbert deformation quantization and the Weyl map (5.3) is a representation of this HDQ.*

Let us have a look on some particular multipliers of the Moyal algebra.

Example 5.2 The symplectic Fourier transform $\mathcal{F} = (\mathcal{F}_L, \mathcal{F}_R)$ is a unitary multiplier in $\mathfrak{M}_b(\mathbf{A}_\theta)$ given by

$$\mathcal{F}_L(f)(x) = \frac{1}{(\pi\theta)^n} \int f(y) e^{\frac{2i}{\theta}\omega(x,y)} dy, \quad \mathcal{F}_R(f)(x) = \frac{1}{(\pi\theta)^n} \int f(y) e^{-\frac{2i}{\theta}\omega(x,y)} dy.$$

The associated automorphism is the change of sign: $U_{\mathcal{F}}(f)(x) := \mathcal{F}_L(\mathcal{F}_R)^*(f)(x) = f(-x)$. We call $\mathcal{F} = (\mathcal{F}_L, \mathcal{F}_R)$ the **Fourier multiplier**. By using the identification of the multipliers with tempered distributions (see Theorem 5.9), we can see that the Fourier multiplier is associated to the Dirac distribution $(\pi\theta)^n\delta(x)$, as also noticed in [28]. \blacklozenge

PROOF First, it is well-known that \mathcal{F}_L and \mathcal{F}_R are unitary (normalization has been chosen for this). Then, for $f_1, f_2 \in L^2(\mathbb{R}^{2n})$, we have

$$f_1 \star_{\theta} \mathcal{F}_L(f_2)(x) = \frac{1}{(\pi\theta)^n} \int f_1(y) f_2(x-y) e^{-\frac{2i}{\theta}\omega(x,y)} dy = \mathcal{F}_R(f_1) \star_{\theta} f_2(x),$$

which shows that $(\mathcal{F}_L, \mathcal{F}_R)$ is a multiplier. \blacksquare

Example 5.3 The **translation** by elements of the Abelian group \mathbb{R}^{2n} :

$$\mathfrak{L}_{x_0}(f)(x) = f(x - x_0)$$

is a group homomorphism $\mathfrak{L} : \mathbb{R}^{2n} \rightarrow \text{Aut}(\mathbf{A}_{\theta})$ valued in the inner automorphisms, due to the \mathbb{R}^{2n} -invariance of the star-product. Indeed,

$$L_{\mathfrak{L}_{x_0}}(f)(x) = f(x - \frac{1}{2}x_0) e^{\frac{i}{\theta}\omega(x_0, x)}, \quad R_{\mathfrak{L}_{x_0}}(f)(x) = f(x + \frac{1}{2}x_0) e^{\frac{i}{\theta}\omega(x_0, x)}$$

is a unitary multiplier and $\mathfrak{L}_{x_0} = L_{\mathfrak{L}_{x_0}}(R_{\mathfrak{L}_{x_0}})^*$. Moreover, \mathfrak{L} induces a group homomorphism $\tilde{\mathfrak{L}} : \mathbb{R}^{2n} \rightarrow \text{Aut}(\mathfrak{M}_b(\mathbf{A}_{\theta}))$. In the identification of Theorem 5.9, \mathfrak{L}_{x_0} is associated to the function $x \mapsto e^{\frac{i}{\theta}\omega(x_0, x)}$. It is easy to see the left action on the multipliers: $\tilde{\mathfrak{L}}_{x_0}(T)(x) = T(x - x_0)$, for any $T \in \mathfrak{M}_b(\mathbf{A}_{\theta})$ seen as a subset of $\mathcal{S}'(\mathbb{R}^{2n})$. \blacklozenge

It turns out that the action of the translation group \mathbb{R}^{2n} , under which the star-product is invariant, is Hamiltonian with respect to the standard symplectic form used for the deformation quantization, and the moment map of the action has the form

$$\forall x, y \in \mathbb{R}^{2n} \quad : \quad \eta_x(y) = \omega(x, y)$$

and it is covariant for the star-product: $[\eta_x, \eta_{x'}]_{\star_{\theta}} = -i\theta\{\eta_x, \eta_{x'}\} = i\theta\omega(x, x')$.

Proposition 5.4 *The Lie algebra $\mathfrak{g} \simeq \mathbb{R}^{2n}$ of the translation group, via the moment map, together with the unit 1, yields an (invariant) symmetry of the HDQ \mathbf{A} . Moreover, the Schwartz HDQ induced by this symmetry coincides with the Schwartz functions: $\mathcal{S}(\mathbf{A}_{\theta}, \mathfrak{g}) = (\mathcal{S}(\mathbb{R}^{2n}), \star_{\theta})$.*

PROOF Left and right \star_{θ} -multiplications by linear functions are unbounded operators with domain containing $\mathcal{D}(\mathbb{R}^{2n})$, and the covariance of \star_{θ} gives the Lie algebra condition on these operators. Moreover, $\mathcal{Z}_b(L^2(\mathbb{R}^{2n})) = \mathbb{C}1$ so it commutes with $\mathcal{U}(\mathfrak{g})$. To identify \mathbf{B} , we have to look at its multiplier topology. By Theorem 4.9, this topology is given by seminorms $\|L_S R_T(f)\|$, where the norm is the L^2 -norm, L_S is the left \star_{θ} -multiplication by the polynomial S and R_T is the right \star_{θ} -multiplication by the polynomial T . Since, the commutator and the anticommutator have the following expression as unbounded operators

$$[x_j, f]_{\star_{\theta}}(x) = i\theta(\omega^{-1}\partial_x)_j f(x), \quad \{x_j, f\}_{\star_{\theta}}(x) = 2x_j f(x),$$

this topology is generated by the seminorms $\|x^{\alpha}\partial^{\beta}f\|$ (α, β multi-indices), and so it corresponds to the standard topology of the Schwartz space by [33]. \blacksquare

It turns out that $\mathcal{S}(\mathbf{A}, \mathfrak{g})$ induced by the translation symmetry is a continuous deformation quantization. Actually, the situation is much better since M. Rieffel proved in [15] that $\mathfrak{M}_b(\mathbf{B})$ is a continuous field of C^* -algebras.

Remark 5.5 The framework of symmetries of HDQ is more general than pullback of group actions on the underlying manifold M . For example on $M = \mathbb{R}^2$ endowed with the Moyal-Weyl product, consider the complex countable-dimensional Lie algebra \mathfrak{g} with generators e^{nx} ($x \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$ or $n \in \mathbb{N}^2$) and satisfying the relations

$$[e^{nx}, e^{mx}]_{\star_\theta} = 2i \sin\left(\frac{\theta}{2} \omega(n, m)\right) e^{(n+m)x}.$$

This symmetry of \mathbf{A} also induces a HDQ of Fréchet algebras. ♦

5.2 Matrix basis and GBV spaces

In this section, we want to show that the GBV spaces $\mathbf{G}^{k,l}(\mathbf{A}_\theta, \mathfrak{g})$ correspond to the ones introduced in [34] with the matrix basis, and that the Sobolev spaces $\mathbf{H}^k(\mathbf{A}, \mathfrak{g})$ correspond to the usual Sobolev spaces $H^k(\mathbb{R}^{2n})$. To get simpler expressions, let us work in dimension 2 ($n = 1$) here, even if the results obtained are valid for arbitrary n .

First, we recall the **matrix basis** $(b_{mn})_{m,n \in \mathbb{N}}$ given in [34] by

$$b_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i(n-m)\varphi} \left(\frac{2r^2}{\theta}\right)^{\frac{n-m}{2}} L_m^{n-m} \left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}},$$

where we use the polar coordinates $x = (x_1, x_2) = (r \cos(\varphi), r \sin(\varphi)) \in \mathbb{R}^2$ and the Laguerre polynomials L_m^k . Such a matrix basis is contained in $\mathcal{S}(\mathbb{R}^2)$ and it satisfies

$$(b_{mn} \star_\theta b_{kl})(x) = \delta_{nk} b_{ml}(x), \quad \int b_{mn}(x) dx = 2\pi\theta \delta_{mn}, \quad \overline{b_{mn}(x)} = b_{nm}(x).$$

The above properties allow to show directly the following result.

Proposition 5.6 *Let $\ell^2(\mathbb{N}^2)$ be the space of infinite matrices $(f_{mn})_{m,n \in \mathbb{N}}$ such that $\sum_{m,n} |f_{mn}|^2 < \infty$. It is a Hilbert algebra for the usual matrix product, the transpose-conjugation and the scalar product associated to the trace. Moreover, there is a Hilbert algebra isomorphism given by*

$$\ell^2(\mathbb{N}^2) \rightarrow (L^2(\mathbb{R}^2), \star_\theta), \quad (f_{mn}) \mapsto f(x) = \sum_{m,n=0}^{\infty} f_{mn} b_{mn}(x)$$

whose inverse has the form

$$f \in L^2(\mathbb{R}^2) \mapsto f_{mn} = \frac{1}{2\pi\theta} \int f(x) b_{nm}(x) dx.$$

The usual **GBV spaces** are defined [34] as

$$G^{k,l}(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2), \sum_{m,n=0}^{\infty} m^k n^l |f_{mn}|^2 < \infty\}.$$

Proposition 5.7 *The GBV spaces $\mathbf{G}^{k,l}(\mathbf{A}_\theta, \mathfrak{g})$ are identical to the usual ones $G^{k,l}(\mathbb{R}^2)$.*

PROOF The translation symmetry \mathfrak{g} contains coordinates x_1, x_2 . We provide a new basis:

$$z_1 := \frac{1}{\sqrt{2\theta}}(x_1 + ix_2), \quad z_2 := \overline{z_1} = \frac{1}{\sqrt{2\theta}}(x_1 - ix_2).$$

The norm $\|\cdot\|_{k,l}$ defining the topology of $\mathbf{G}^{k,l}(\mathbf{A}_\theta, \mathfrak{g})$ in Definition 4.12 is expressed in terms of $x_1, x_2 \in \mathfrak{g}$, but it can be equivalently reformulated in terms of z_1, z_2 . Moreover, the isomorphism of Proposition 5.6 can be extended to the polynomials and we have

$$z_1 \mapsto i\sqrt{m} \delta_{m,n+1}, \quad z_2 \mapsto -i\sqrt{m+1} \delta_{m+1,n}.$$

Therefore, we can compute the norm in terms of the matrix basis coefficients:

$$\|z_{i_1} \star_\theta \cdots \star_\theta z_{i_p} \star_\theta f \star_\theta z_{j_q} \star_\theta \cdots \star_\theta z_{j_1}\|^2 = \sum_{m,n=0}^{\infty} (m + \alpha_1) \cdots (m + \alpha_p)(n + \beta_1) \cdots (n + \beta_q) |f_{mn}|^2,$$

where α_a, β_b are real constants depending on the indices i_1, \dots, i_p and j_1, \dots, j_q , and satisfying $|\alpha_a| \leq p, |\beta_q| \leq q$. With this expression, we see immediatly that $\mathbf{G}^{k,l}(\mathbf{A}_\theta, \mathfrak{g}) = G^{k,l}(\mathbb{R}^2)$. ■

Using the expression of the commutator $[x_j, \cdot]_{\star_\theta} = i\theta(\omega \partial_x)_j$, we obtain the following result concerning the Sobolev spaces.

Proposition 5.8 *The Sobolev spaces $\mathbf{H}^k(\mathbf{A}_\theta, \mathfrak{g})$ are identical to the usual ones $H^k(\mathbb{R}^2)$.*

5.3 Link with distributions

Let us try to identify the multiplier space $\mathfrak{M}(\mathbf{B})$ of $\mathbf{B} = \mathcal{S}(\mathbb{R}^{2n})$. We will show here that this unbounded notion of multipliers of Hilbert algebra corresponds in this particular case to the well-known notion of multipliers of a Fréchet algebra.

Recall that for $(\mathbf{B}_\theta, \star_\theta)$ be an arbitrary Hilbert subalgebra of $\mathbf{A}_\theta = (L^2(\mathbb{R}^{2n}), \star_\theta)$ and a Fréchet algebra containing $\mathcal{D}(M)$ and contained in $\mathcal{C}^\infty(M)$, we can also define a notion of Fréchet multiplier as following. By denoting also $\langle -, - \rangle$ the duality bracket between the distributions \mathbf{B}'_θ and the functions \mathbf{A}_θ , the product \star_θ satisfying a tracial identity can be extended as

$$\forall T \in \mathbf{B}'_\theta, \forall f, h \in \mathbf{B}_\theta \quad : \quad \langle T \star_\theta f, h \rangle := \langle T, f \star_\theta h \rangle \text{ and } \langle f \star_\theta T, h \rangle := \langle T, h \star_\theta f \rangle, \quad (5.4)$$

which is compatible with the case $T \in \mathbf{B}_\theta$ (the duality bracket corresponds then to the scalar product). Then, the multiplier space associated to the Fréchet algebra \mathbf{B}_θ has the form

$$\mathcal{M}_{\star_\theta}(\mathbf{B}_\theta) := \{T \in \mathbf{B}'_\theta, f \mapsto T \star_\theta f \text{ and } f \mapsto f \star_\theta T \text{ are continuous from } \mathbf{B}_\theta \text{ into itself}\}.$$

This space is equipped with the topology associated to the seminorms:

$$\|T\|_{B,j,L} = \sup_{f \in B} \|T \star_\theta f\|_j \text{ and } \|T\|_{B,j,R} = \sup_{f \in B} \|f \star_\theta T\|_j$$

where B is a bounded subset of \mathbf{B}_θ and $\|f\|_j$ are the Fréchet seminorms of \mathbf{B}_θ . The product \star_θ can be extended to $\mathcal{M}_{\star_\theta}(\mathbf{B}_\theta)$ by:

$$\forall S, T \in \mathcal{M}_{\star_\theta}(\mathbf{B}_\theta), \forall f \in \mathbf{B}_\theta \quad : \quad \langle S \star_\theta T, f \rangle := \langle S, T \star_\theta f \rangle = \langle T, f \star_\theta S \rangle.$$

And $(\mathcal{M}_{\star_\theta}(\mathbf{B}_\theta), \star_\theta)$ is an associative Hausdorff locally convex algebra, called the **Fréchet multiplier algebra**.

Theorem 5.9 *Let $(\mathbf{B}_\theta, \star_\theta)$ be a Hilbert subalgebra of \mathbf{A}_θ as well as a nuclear Fréchet algebra containing $\mathcal{D}(M)$ and contained in $\mathcal{C}^\infty(M)$. Then, $\mathcal{M}_{\star_\theta}(\mathbf{B}_\theta) \simeq \mathfrak{M}(\mathbf{B}_\theta)$.*

PROOF We will use the following inclusion $\mathbf{B}_\theta \hookrightarrow \mathbf{B}'_\theta$ to see first $\mathfrak{M}(\mathbf{B}_\theta)$ as a subspace of the distributions \mathbf{B}'_θ . Let us consider an arbitrary $T = (L, R) \in \mathfrak{M}(\mathbf{B}_\theta)$. Due to Proposition 3.32, the maps $L, R : \mathbf{B}_\theta \rightarrow \mathbf{B}'_\theta$ are continuous for the Fréchet topologies. Due to Schwartz kernel's theorem, there exist kernels $K_L, K_R \in \mathbf{B}'_\theta \hat{\otimes} \mathbf{B}'_\theta$ such that $\forall f \in \mathbf{B}_\theta, \forall x \in M$,

$$L(f)(x) = \int K_L(x, y) f(y) dy, \quad R(f)(x) = \int K_R(x, y) f(y) dy,$$

where we use by convenience the symbol integral to mean only the duality bracket for distributions (extending the scalar product of \mathbf{A}_θ). Implementing the conditions of L to be a left multiplier leads to

$$\begin{aligned} \frac{1}{(\pi\theta)^{2n}} \int K_L(x, y-z) f_1(y) \mathcal{F}_R(f_2)(z) e^{-\frac{2i}{\theta}\omega(y,z)} dy dz &= L(f_1 \star_\theta f_2)(x) \\ &= L(f_1) \star_\theta f_2(x) = \frac{1}{(\pi\theta)^{2n}} \int K_L(x+z, y) f_1(y) \mathcal{F}_R(f_2)(z) e^{-\frac{2i}{\theta}\omega(x,z)} dy dz, \end{aligned}$$

for any $f_1, f_2 \in \mathbf{B}_\theta$ (dense in \mathbf{A}_θ). This is equivalent to the condition

$$K_L(x, y) = K_L(x-y, 0) e^{\frac{2i}{\theta}\omega(x,y)}.$$

In the same way, we have $K_R(x, y) = K_R(x-y, 0) e^{-\frac{2i}{\theta}\omega(x,y)}$. We define the following distributions in \mathbf{B}'_θ :

$$k_L := (\pi\theta)^n \mathcal{F}_R(K_L(\cdot, 0)), \quad k_R := (\pi\theta)^n \mathcal{F}_L(K_R(\cdot, 0)).$$

A simple computation gives that $\forall f \in \mathbf{B}_\theta$,

$$L(f)(x) = \frac{1}{(\pi\theta)^{2n}} \int k_L(y) f(z) e^{-\frac{2i}{\theta}(\omega(y,z) + \omega(z,x) + \omega(x,y))} dy dz =: (k_L \star_\theta f)(x),$$

where \star_θ is defined between \mathbf{B}'_θ and \mathbf{B}_θ by (5.4), i.e. the integral has a distributional meaning. We have also $R(f)(x) = (f \star_\theta k_R)(x)$. Now, the condition $f_1 \star_\theta L(f_2) = R(f_1) \star_\theta f_2$ of (L, R) being a multiplier implies that $k_L = k_R$. Since $(L, R) \in \mathfrak{M}(\mathbf{B}_\theta)$, we have $k_L = k_R \in \mathcal{M}_{\star_\theta}(\mathbf{B}_\theta)$ due to its definition. It is then easy to check that

$$\begin{aligned} k_{(L_1 L_2)}(x) &= \frac{1}{(\pi\theta)^{2n}} \int k_{L_1}(y) k_{L_2}(z) e^{-\frac{2i}{\theta}(\omega(y,z) + \omega(z,x) + \omega(x,y))} dy dz = (k_{L_1} \star_\theta k_{L_2})(x), \\ k_{(L^*)}(x) &= \overline{k_L(x)}, \end{aligned}$$

as distributions, so that the map $(L, R) \in \mathfrak{M}(\mathbf{B}_\theta) \mapsto k_L = k_R \in \mathcal{M}_{\star_\theta}(\mathbf{B}_\theta)$ is a \ast -algebra morphism. The map $k \in \mathcal{M}_{\star_\theta}(\mathbf{B}_\theta) \mapsto (k \star_\theta \cdot, \cdot \star_\theta k) \in \mathfrak{M}(\mathbf{B}_\theta)$ is obviously an inverse map. \blacksquare

We can now apply this theorem directly to the algebra $\mathbf{B}_\theta = (\mathcal{S}(\mathbb{R}^{2n}), \star_\theta)$ and we see that its unbounded multiplier algebra $\mathfrak{M}(\mathbf{B}_\theta)$ corresponds to the Fréchet multipliers of $\mathcal{S}(\mathbb{R}^{2n})$ (a subspace of the tempered distributions $\mathcal{S}'(\mathbb{R}^{2n})$) used in [32, 34, 28].

Remark 5.10 The Weyl map (5.3) restricted to $\mathcal{S}(\mathbf{A}, \mathfrak{g})$ is extendable. Due to Theorem 4.9 and since $\mathcal{D}_{\mathcal{S}(\mathbf{A}, \mathfrak{g})} = \mathcal{S}(\mathbb{R}^n)$, we obtain a faithful \ast -representation

$$\tilde{\Omega}_\theta : \mathcal{M}_{\star_\theta}(\mathcal{S}(\mathbb{R}^{2n})) \rightarrow \mathcal{L}^+(\mathcal{S}(\mathbb{R}^n))$$

that was already considered in [32] for the Moyal-Weyl quantization. \blacklozenge

5.4 Infinite-dimensional Clifford algebras

As an illustration of the unital case, let us consider the well-known hyperfinite type II_1 factor, but seen as a limit of deformation quantization.

Let V be an infinite dimensional separable real vector space with a positive definite scalar product. We consider an orthonormal basis $(\xi_i)_{i \in \mathbb{N}}$ of V . The Clifford algebra $Cl(V)$ consists in the tensor algebra of the complexification of V quotiented by the ideal generated by $\{v \otimes v - \langle v, v \rangle \mathbb{1}\}$. It is generated by the ξ_i satisfying

$$\xi_i \xi_j + \xi_j \xi_i = 2\delta_{ij} \mathbb{1}.$$

A basis of V is given by the $\xi_I := \prod_{i \in I} \xi_i$ where the product is ordered, $\xi_\emptyset = \mathbb{1}$ and I are subsets of \mathbb{N} .

The following is a consequence of [35]. The algebra $Cl(V)$ has an involution as well as a normalized ($\tau(\mathbb{1}) = 1$) hermitian ($\tau(x^*) = \overline{\tau(x)}$) trace ($\tau(xy) = \tau(yx)$) $\tau : Cl(V) \rightarrow \mathbb{C}$ defined by

$$x^* := \sum_I (-1)^{\frac{1}{2}|I|(|I|-1)} \overline{x_I} \xi_I, \quad \tau(x) = x_\emptyset,$$

for any $x = \sum_I x_I \xi_I$ with $x_I \in \mathbb{C}$, $I \subset \mathbb{N}$ and the sum is finite. The map $(x, y) \mapsto \tau(x^*y)$ is a sesquilinear hermitian positive definite form and we note \mathcal{H} the completion of $Cl(V)$ for the norm associated to this sesquilinear form.

Then, $Cl(V)$ acts by left-multiplication on \mathcal{H} and we note $Cl[V]$ the weak completion of $Cl(V)$ seen as a subalgebra of $\mathcal{L}(\mathcal{H})$. $Cl[V]$ is the hyperfinite factor of type II_1 and τ extends to $Cl[V]$ as a finite faithful normal trace. Since $Cl(V)$ coincide with the Moyal deformation quantization [21, 36] of the superspace $\mathbb{R}^{0|2m}$ if $2m = \dim(V) < +\infty$, and since $\bigcup_{m=1}^\infty Cl(2m)$ is dense in $Cl[V]$, we can see $Cl[V]$ as the limit of the deformation quantization of $\mathbb{R}^{0|2m}$ when $m \rightarrow \infty$. Taking this into account, we have now that $Cl(V)$ is a Hilbert algebra of completion \mathcal{H} . Due to Proposition 3.35, we have the following result.

Proposition 5.11 *The bounded multipliers $\mathfrak{M}_b(\mathbf{A})$ of $\mathbf{A} := Cl(V)$ are left and right multiplication by elements of $Cl[V]$. Moreover, the unbounded multipliers are $\mathfrak{M}(\mathbf{A}) = Cl(V)$ that is a non-closed O^* -algebra.*

Note that the hyperfactor of type II_∞ can be obtained as the tensor product $\mathfrak{M}_b(\mathbb{R}^{2n}) \overline{\otimes} Cl[V]$, so as the limit of the deformation quantization $\mathbb{R}^{2n|2m}$ with $m \rightarrow \infty$.

5.5 Deformation quantization of normal j -groups

To show the efficiency of this framework of Hilbert deformation quantization, we look at another deformation quantization that was defined and studied in [25]. Let us first describe what are **elementary normal j -groups** (see [26]). They are AN Iwasawa factor of the simple Lie groups $SU(1, n)$ for $n \in \mathbb{N}^*$. Explicitly they are realized as $\mathbb{S} = \mathbb{R} \times V \times \mathbb{R}$, where (V, ω) is a symplectic vector space of dimension $2n$. With the coordinate system (a, x, ℓ) associated to such a realization, the group law of \mathbb{S} has the form

$$(a, x, \ell) \cdot (a', x', \ell') = \left(a + a', e^{-a'} x + x', e^{-2a'} \ell + \ell' + \frac{1}{2} e^{-a'} \omega(x, x') \right) \quad (5.5)$$

and the inverse $(a, x, \ell)^{-1} = (-a, -e^a x, -e^{2a} \ell)$. It turns out that the generic coadjoint orbit of this group is \mathbb{S} -equivariantly diffeomorphic to the Lie group \mathbb{S} itself and under this identification the KKS symplectic form has the expression $\omega_{\mathbb{S}} = 2da \wedge d\ell + \omega$. The coadjoint action (or left-multiplication under the identification) has the associated moment maps

$$\eta_H(a, x, \ell) = 2\ell, \quad \eta_y(a, x, \ell) = e^{-a} \omega(y, x), \quad \eta_E(a, x, \ell) = e^{-2a}, \quad (5.6)$$

with decomposition of the Lie algebra $\mathfrak{s} := \mathbb{R}H \oplus V \oplus \mathbb{R}E$ and exponential map $(a, x, \ell) = e^{aH}e^xe^{\ell E}$.

On such groups, P. Bieliavsky and M. Massar introduced in [24] an associative \mathbb{S} -invariant star-product that has the following form

$$(f_1 \star_\theta f_2)(g) = \frac{1}{(\pi\theta)^{2n+2}} \int A_{\mathbb{S}}(g, g_1, g_2) e^{-\frac{2i}{\theta} S_{\mathbb{S}}(g, g_1, g_2)} f_1(g_1) f_2(g_2) dg_1 dg_2 \quad (5.7)$$

for $f_1, f_2 \in \mathcal{D}(\mathbb{S})$, $g_i := (a_i, x_i, \ell_i) \in \mathbb{S}$, the left Haar measure $dg := dadxd\ell$, and where the amplitude and the phase are

$$\begin{aligned} A_{\mathbb{S}}(g, g_1, g_2) &= 4\sqrt{\cosh(2(a_1 - a_2)) \cosh(2(a_1 - a)) \cosh(2(a - a_2))} \cosh(a_2 - a)^n \\ &\quad \cosh(a_1 - a)^n \cosh(a_1 - a_2)^n, \\ S_{\mathbb{S}}(g, g_1, g_2) &= -\sinh(2(a_1 - a_2))\ell - \sinh(2(a_2 - a))\ell_1 - \sinh(2(a - a_1))\ell_2 \\ &\quad + \cosh(a_1 - a) \cosh(a_2 - a) \omega(x_1, x_2) + \cosh(a_1 - a) \cosh(a_1 - a_2) \omega(x_2, x) \\ &\quad + \cosh(a_1 - a_2) \cosh(a_2 - a) \omega(x, x_1). \end{aligned}$$

This product, which extends to $L^2(\mathbb{S})$, is related to the Moyal-Weyl product (5.1) (but on \mathbb{R}^{2n+2} and for the symplectic form $\omega_{\mathbb{S}}$ instead of ω), that we denote \star_θ^0 in this section, via an intertwining operator U_θ : $f_1 \star_\theta f_2 = U_\theta((U_\theta^{-1} f_1) \star_\theta^0 (U_\theta^{-1} f_2))$ for $f_1, f_2 \in \mathcal{D}(\mathbb{S})$. These operators have the form

$$\begin{aligned} U_\theta f(a, x, \ell) &= \frac{1}{2\pi} \int dt d\xi \sqrt{\cosh(\frac{\theta t}{2})} \cosh(\frac{\theta t}{4})^n e^{\frac{2i}{\theta} \sinh(\frac{\theta t}{2})\ell - i\xi t} f(a, \cosh(\frac{\theta t}{4})x, \xi), \\ U_\theta^{-1} f(a, x, \ell) &= \frac{1}{2\pi} \int dt d\xi \frac{\sqrt{\cosh(\frac{\theta t}{2})}}{\cosh(\frac{\theta t}{4})^n} e^{-\frac{2i}{\theta} \sinh(\frac{\theta t}{2})\xi + it\ell} f(a, \cosh(\frac{\theta t}{4})^{-1}x, \xi). \end{aligned} \quad (5.8)$$

Finally the product (5.7) is associated to the following quantization map [25]

$$\begin{aligned} \Omega_\theta(f) \varphi(a_0, v_0) &:= \frac{2}{(\pi\theta)^{n+1}} \int_{\mathbb{S}} f(a, v, w, \ell) \sqrt{\cosh(2(a - a_0))} \cosh(a - a_0)^n \\ &\quad e^{\frac{2i}{\theta} \left(\sinh(2(a - a_0))\ell + \omega(\cosh(a - a_0)v - v_0, \cosh(a - a_0)w) \right)} \varphi(2a - a_0, 2 \cosh(a - a_0)v - v_0) dadvdwd\ell, \end{aligned} \quad (5.9)$$

for $f \in \mathcal{D}(\mathbb{S})$, $\varphi \in L^2(Q)$, $Q := \mathbb{R}H \times V_0$, V_0 a Lagrangian subspace of V , $(v, w) \in V$ in the integral, and $(a_0, v_0) \in Q$.

In [25], a modified Schwartz space was introduced on \mathbb{S} . Let us recall its definition. The left-invariant vector fields of \mathbb{S} are given by

$$\tilde{H} = \partial_a - x\partial_x - 2\ell\partial_\ell, \quad \tilde{y} = y\partial_x + \frac{1}{2}\omega(x, y)\partial_\ell, \quad \tilde{E} = \partial_\ell,$$

with H , y and E generators of the Lie algebra \mathfrak{s} . The maps α are

$$\alpha_H(g) = \ell, \quad \alpha_y(g) = \cosh(a)\omega(y, x), \quad \alpha_E(g) = \sinh(2a),$$

for any $g = (a, x, \ell) \in \mathbb{S}$. This leads to the following definition. The **modified Schwartz space** of \mathbb{S} is defined as

$$\mathcal{S}(\mathbb{S}) = \{f \in C^\infty(\mathbb{S}) \mid \forall j \in \mathbb{N}^{2n+2}, \forall P \in \mathcal{U}(\mathfrak{s}) \quad \|f\|_{j,P} := \sup_{g \in \mathbb{S}} |\alpha^j(g) \tilde{P}f(g)| < \infty\} \quad (5.10)$$

where $\alpha^j := \alpha_H^{j_1} \alpha_{e_1}^{j_2} \dots \alpha_{e_{2n}}^{j_{2n+1}} \alpha_E^{j_{2n+2}}$. It is a Fréchet nuclear algebra endowed with the seminorms $(\|f\|_{j,P})$ and the star-product (5.7).

Let us see how the above setting fits in the formalism of HDQ. Denote $\mathbf{A}_\theta := L^2(\mathbb{S})$ for $\theta \neq 0$ and $\mathbf{A}_0 := L^2(\mathbb{S}) \cap L^\infty(\mathbb{S})$.

Proposition 5.12 *The family $\mathbf{A} := (\mathbf{A}_\theta)$ defines a Hilbert deformation quantization and the operator (5.8) is an intertwiner (in the sense of Definition 4.1) between the Moyal-Weyl HDQ $(L^2(\mathbb{R}^{2n+2}), \star_\theta^0)$ and \mathbf{A} . Moreover, the quantization map (5.9) is related to the Weyl map (5.3), denoted now by Ω_θ^0 , in the following way: $\Omega_\theta = \Omega_\theta^0 \circ U_\theta^{-1}$, and it is a representation of \mathbf{A} .*

PROOF First, we know from [24] that U_θ is an algebra homomorphism. It is easy to see that it is also compatible with complex conjugation and scalar products:

$$\begin{aligned} \int |U_\theta f(a, x, \ell)|^2 da dx d\ell &= \frac{1}{2\pi} \int \cosh\left(\frac{\theta t}{4}\right)^{2n} e^{i(\xi - \xi')t} \overline{f(a, \cosh(\frac{\theta t}{4})x, \xi)} f(a, \cosh(\frac{\theta t}{4})x, \xi') da dx dt d\xi d\xi' \\ &= \int |f(a, x, \xi)|^2 da dx d\xi, \end{aligned}$$

first by integrating on ℓ , then by changing the variable x and integrating over t . So U_θ is an isometric isomorphism from $(L^2(\mathbb{R}^{2n+2}), \star_\theta^0)$ and $(L^2(\mathbb{S}), \star_\theta)$, which shows that $(L^2(\mathbb{S}), \star_\theta)$ is a complete Hilbert algebra. Actually, it was already directly proved in [25] that this was a Hilbert algebra. Then, a direct computation shows the relation $\Omega_\theta = \Omega_\theta^0 \circ U_\theta^{-1}$. ■

Corollary 5.13 $\mathfrak{M}_b(\mathbf{A})$ is a continuous field of C^* -algebras.

PROOF This is a direct consequence of the fact that $\mathfrak{M}_b(L^2(\mathbb{R}^{2n+2}), \star_\theta)$ is a continuous field of C^* -algebras [15], of the fact that $U : L^2(\mathbb{R}^{2n+2}) \rightarrow L^2(\mathbb{S})$ defined by (5.8) is an intertwiner and of Proposition 4.4. ■

Example 5.14 Let us define here a particular multiplier of \mathbf{A}_θ that gives a new **Fourier transformation**. The symplectic Fourier transformation is a multiplier of the Moyal-Weyl HDQ associated to the distribution $(\pi\theta)^n \delta(x)$ (see Example 5.2). Let us push such a Dirac distribution by the intertwiner U_θ :

$$\delta_{\mathbb{S}}(a, x, \ell) = (\pi\theta)^{n+1} U_\theta(\delta)(a, x, \ell) = (\pi\theta)^n \delta(a) \delta(x) \int (1+t^2)^{-\frac{1}{4}} c(t)^{-n} e^{\frac{2i}{\theta} t \ell} dt$$

Then, this Fourier transform $\mathcal{F}_{\mathbb{S}} = (\mathcal{F}_{\mathbb{S},L}, \mathcal{F}_{\mathbb{S},R})$ is the unitary multiplier in $\mathfrak{M}_b(\mathbf{A}_\theta)$ associated to $\delta_{\mathbb{S}}$, namely

$$\begin{aligned} \mathcal{F}_{\mathbb{S},L}(f)(a, x, \ell) &= \frac{4}{(\pi\theta)^{n+1}} \int \sqrt{\cosh(2a) \cosh(2a')} \cosh(a)^n \cosh(a')^n f(a', x', \ell') \\ &\quad e^{-\frac{2i}{\theta} (\sinh(2a')\ell - \sinh(2a)\ell' + \cosh(a) \cosh(a') \omega(x', x))} da' dx' d\ell', \\ \mathcal{F}_{\mathbb{S},R}(f)(a, x, \ell) &= \frac{4}{(\pi\theta)^{n+1}} \int \sqrt{\cosh(2a) \cosh(2a')} \cosh(a)^n \cosh(a')^n f(a', x', \ell') \\ &\quad e^{+\frac{2i}{\theta} (\sinh(2a')\ell - \sinh(2a)\ell' + \cosh(a) \cosh(a') \omega(x', x))} da' dx' d\ell'. \end{aligned}$$

We have therefore defined two unitary transformations on \mathbf{A}_θ . The associated automorphism is also the change of sign: $U_{\mathcal{F}_{\mathbb{S}}}(f)(a, x, \ell) := \mathcal{F}_{\mathbb{S},L}(\mathcal{F}_{\mathbb{S},R})^*(f)(a, x, \ell) = f(-a, -x, -\ell)$. ♦

5.6 Symmetry for normal j-groups

Let us now identify the space $\mathcal{S}(\mathbb{S})$ in the framework of HDQ.

Theorem 5.15 *The subspace of $\mathcal{C}^\infty(\mathbb{S})$ generated by the functions ℓ , constants, $e^{-a}x_j$, $e^a x_j$ ($j \in \{1, \dots, 2n\}$), e^{-2a} and e^{2a} in the coordinate system (a, x, ℓ) of \mathbb{S} forms a symmetry \mathfrak{g} of the HDQ \mathbf{A} of Proposition 5.12. The Schwartz HDQ $\mathbf{B} := \mathbf{S}(\mathbf{A}, \mathfrak{g})$ induced by this symmetry corresponds to the modified Schwartz space $\mathcal{S}(\mathbb{S})$ defined in (5.10).*

It turns out that \mathfrak{g} is a representation of the Lie algebra of the transvection group of the bounded symmetric domain associated (and isomorphic) to \mathbb{S} , so that we can call this symmetry \mathfrak{g} the **transvection symmetry**. The action of this transvection group is actually an invariant symmetry of the HDQ.

PROOF First, the above generators form a symmetry of \mathbf{A} with Lie relations

$$\begin{aligned} [\ell, e^{-2a}]_{\star_\theta} &= -i\theta e^{-2a}, & [\ell, e^{2a}]_{\star_\theta} &= i\theta e^{2a}, & [e^{\varepsilon a}\omega(y, x), e^{\varepsilon' a}\omega(y', x)]_{\star_\theta} &= -i\theta e^{(\varepsilon+\varepsilon')a}\omega(y, y'), \\ [\ell, e^{-a}\omega(y, x)]_{\star_\theta} &= -\frac{i\theta}{2}e^{-a}\omega(y, x), & [\ell, e^a\omega(y, x)]_{\star_\theta} &= \frac{i\theta}{2}e^a\omega(y, x), \end{aligned}$$

for $\varepsilon, \varepsilon' = \pm 1$. Let us show that the multiplier topology of \mathbf{B}_θ , i.e. the one generated by the seminorms $\|L_S R_T f\|$, for $f \in \mathbf{B}_\theta$, $S, T \in \mathcal{U}(\mathfrak{g})$, corresponds to the topology of $\mathcal{S}(\mathbb{S})$, i.e. the one generated by the seminorms $\|\sinh(2a)^{k_1} x^\alpha \ell^{k_2} \partial_a^{k_3} \partial_x^\beta \partial_\ell^{k_4} f\|_\infty$, for $k_i \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^{2n}$. It turns out that the part of the multiplier topology corresponding to the generator ℓ coincides obviously with the part of the topology of $\mathcal{S}(\mathbb{S})$ corresponding to the operators $\ell^{k_2} \partial_a^{k_3}$, as in the case of the Moyal product. Let us compare the part of the multiplier topology corresponding to the generators e^{-2a}, e^{2a} with the part of the topology of $\mathcal{S}(\mathbb{S})$ corresponding to the operators $\sinh(2a)^{k_1} \partial_\ell^{k_4}$.

A direct computation using the explicit expression (5.7) shows that

$$\begin{aligned} [e^{-2ka}, f]_{\star_\theta}(a, x, \ell) &= -\frac{2}{\pi\theta} e^{-2ka} \int_{\mathbb{R}^2} \sinh(k \cdot \operatorname{arcsinh}(u)) e^{\frac{2i}{\theta} u(\ell-v)} f(a, x, v) du dv, \\ \{e^{-2ka}, f\}_{\star_\theta}(a, x, \ell) &= \frac{2}{\pi\theta} e^{-2ka} \int_{\mathbb{R}^2} \cosh(k \cdot \operatorname{arcsinh}(u)) e^{\frac{2i}{\theta} u(\ell-v)} f(a, x, v) du dv, \end{aligned}$$

where the anticommutator is denoted as $\{f, g\}_{\star_\theta} = f \star_\theta g + g \star_\theta f$.

- If k is even, $\cosh(k \cdot \operatorname{arcsinh}(u)) = P_k(u) \varepsilon_0(u)$, with P_k a real polynomial of degree k and $\varepsilon_0(u) = 1$.
- If k is odd, $\cosh(k \cdot \operatorname{arcsinh}(u)) = P_k(u) \varepsilon_1(u)$, with P_k a real polynomial of degree $k-1$ and $\varepsilon_1(u) = \sqrt{1+u^2}$.
- If k is even, $\sinh(k \cdot \operatorname{arcsinh}(u)) = Q_k(u) \varepsilon_1(u)$, with Q_k a real polynomial of degree $k-1$ and $\varepsilon_1(u) = \sqrt{1+u^2}$.
- If k is odd, $\sinh(k \cdot \operatorname{arcsinh}(u)) = Q_k(u) \varepsilon_0(u)$, with Q_k a real polynomial of degree k and $\varepsilon_0(u) = 1$.

In particular, $P_1(u) = 1$, $Q_1(u) = u$, $P_2(u) = 1 + 2u^2$, $Q_2(u) = 2u$, $P_3(u) = 1 + 4u^2$, $Q_3(u) = 3u + 4u^3, \dots$ We can now compute the L^2 -norms of these quantities. For example,

$$\begin{aligned} \|[e^{-2ka}, f]_{\star_\theta}\|^2 &= \frac{4}{\pi\theta} \int Q_k(u)^2 \varepsilon_{k+1}(u)^2 e^{-4ka} e^{-\frac{2i}{\theta} u(v'-v)} \overline{f(a, x, v)} f(a, x, v'), da du dv dv' dx \\ &= \frac{4}{\pi\theta} \int e^{-4ka} e^{-\frac{2i}{\theta} u(v'-v)} \overline{f(a, x, v)} (Q_k \varepsilon_{k+1})^2 \left(\frac{-i\theta}{2} \partial_{v'}\right) f(a, x, v'), da du dv dv' dx \end{aligned}$$

by using integration by parts since $(Q_k \varepsilon_{k+1})^2$ is always a polynomial. We get

- If k is even, $\|\{e^{-2ka}, f\}_{\star_\theta}\|^2 = 4\|e^{-2ka}P_k(\frac{-i\theta}{2}\partial_\ell)f\|^2$.
- If k is odd, $\|\{e^{-2ka}, f\}_{\star_\theta}\|^2 = 4\|e^{-2ka}P_k(\frac{-i\theta}{2}\partial_\ell)f\|^2 + \theta^2\|e^{-2ka}P_k(\frac{-i\theta}{2}\partial_\ell)\partial_\ell f\|^2$.
- If k is even, $\|[e^{-2ka}, f]_{\star_\theta}\|^2 = 4\|e^{-2ka}Q_k(\frac{-i\theta}{2}\partial_\ell)f\|^2 + \theta^2\|e^{-2ka}Q_k(\frac{-i\theta}{2}\partial_\ell)\partial_\ell f\|^2$.
- If k is odd, $\|[e^{-2ka}, f]_{\star_\theta}\|^2 = 4\|e^{-2ka}Q_k(\frac{-i\theta}{2}\partial_\ell)f\|^2$.

Since the star-multiplication of powers of e^{-2a} and e^{2a} is just the pointwise multiplication, we see that the a -part of the multiplier topology of \mathbf{B}_θ is generated by the seminorms $\|\{e^{-2ka}, f\}_{\star_\theta}\|$ and $\|[e^{-2ka}, f]_{\star_\theta}\|$. And these are equivalent to the seminorms $\|e^{-2k_1a}\partial_\ell^{k_2}f\|$ (for the use of the L^2 -norm), and also to the seminorms $\|\sinh(2a)^{k_1}\partial_\ell^{k_2}f\|$ (by using the inequality $\cosh(2a) \leq 1 + \sinh(2a)^2$).

What remains to be done for this part concerning the generators $e^{\pm 2a}$ is the equivalence between these seminorms and the seminorms using the same operators but with the L^∞ -norm. We adapt the standard argument of [33] and we concentrate on the variable a and look at the low order case. Since $(1 + \sinh(2a)^2)^{-1}$ is in $L^2(\mathbb{R})$, then

$$\|f\|_2 \leq \|(1 + \sinh(2a)^2)^{-1}\|_2 \|(1 + \sinh(2a)^2)f\|_\infty \leq C(\|f\|_\infty + \|\sinh(2a)^2f\|_\infty).$$

For the other sense of the equivalence, we use $f(a) = \int_{-\infty}^a \partial_{a'} f(a') da'$ and we obtain

$$\|f\|_\infty \leq \|\partial_a f\|_1 \leq \|(1 + \sinh(2a)^2)^{-1}\|_2 \|(1 + \sinh(2a)^2)\partial_a f\|_2$$

by using also the Cauchy-Schwartz inequality.

Now let us compare the part of the multiplier topology corresponding to the generator x with the part of the topology of $\mathcal{S}(\mathbb{S})$ corresponding to the operators $x^\alpha \partial_x^\beta$. The philosophy is the same as before so let us do it for degree 1. First, a computation gives the following expressions (for $\varepsilon = \pm 1$):

$$\begin{aligned} [e^{\varepsilon a} \omega(y, x), f]_{\star_\theta} &= i\theta e^{\varepsilon a} (y \partial_x - \frac{\varepsilon}{2} \omega(y, x) \partial_\ell) f(a, x, \ell) \\ \{e^{\varepsilon a} \omega(y, x), f\}_{\star_\theta} &= \frac{e^{\varepsilon a}}{\pi \theta} \int e^{-\frac{2i}{\theta} t(s-\ell)} \left(2c(t)^2 \omega(y, x) + i\theta \varepsilon \frac{s(t)}{c(t)} y \partial_x \right) f(a, x, s) dt ds \end{aligned}$$

with $c(t) := \cosh(\frac{1}{2} \operatorname{arcsinh}(t)) = \left(\frac{1 + \sqrt{1+t^2}}{2} \right)^{\frac{1}{2}}$ and $s(t) := \sinh(\frac{1}{2} \operatorname{arcsinh}(t))$.

We obtain directly the inequality $\|e^{\varepsilon a} \omega(y, x), f\|_{\star_\theta} \leq \theta \|\sinh(2a) y \partial_x f\| + \frac{\theta}{2} \|\sinh(2a) \omega(y, x) \partial_\ell f\|$ for the commutator. For the anticommutator, notice that $\mathcal{F}^{-1} T \mathcal{F}$ is a positive operator as soon as T is a positive operator, where \mathcal{F} denotes here the usual Fourier transform. Then we use the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for the commuting selfadjoint operators $\omega(y, x)$ and $iy \partial_x$:

$$\left(2c(t)^2 \omega(y, x) - i\theta \frac{s(t)}{c(t)} y \partial_x \right)^2 \leq 2 \left(4c(t)^4 \omega(y, x)^2 - \theta^2 \frac{s(t)^2}{c(t)^2} (y \partial_x)^2 \right)$$

and we obtain for the anticommutator

$$\|e^{\varepsilon a} \omega(y, x), f\|_{\star_\theta}^2 \leq 8\|e^{\varepsilon a} \omega(y, x) f\|^2 + 2\theta^2 \|e^{\varepsilon a} \omega(y, x) \partial_\ell f\|^2 + \frac{\theta^4}{8} \|e^{\varepsilon a} y \partial_x \partial_\ell f\|^2.$$

by using integrations by parts and transforming t in $\frac{i\theta}{2} \partial_s$. This means that the multiplier topology is controlled by the one of $\mathcal{S}(\mathbb{S})$ (we have also the equivalence between L^2 -norms and

L^∞ -seminorms for these operators). For the converse sense, concerning the operators $x^\alpha \partial_x^\beta$, we have $y \partial_x f = \frac{i}{\theta} (e^a [e^{-a} \omega(y, x), f]_{\star_\theta} - e^{-a} [e^a \omega(y, x), f]_{\star_\theta})$ so that

$$\|y \partial_x f\| \leq \frac{1}{\theta} \|\cosh(2a) [e^{-a} \omega(y, x), f]_{\star_\theta}\| + \frac{1}{\theta} \|\cosh(2a) [e^a \omega(y, x), f]_{\star_\theta}\|$$

and we know already that $\|\cosh(2a) f\|$ is controlled by the multiplier topology. Since we have

$$\begin{aligned} \left(2c(t)^2 \omega(y, x) - i\theta \frac{s(t)}{c(t)} y \partial_x\right)^2 + \left(2c(t)^2 \omega(y, x) + i\theta \frac{s(t)}{c(t)} y \partial_x\right)^2 \\ = 2 \left(4c(t)^4 \omega(y, x)^2 - \theta^2 \frac{s(t)^2}{c(t)^2} (y \partial_x)^2\right) \geq 8\omega(y, x)^2, \end{aligned}$$

by using the above computations and arguments, we find that

$$\|\omega(y, x) f\|^2 \leq \frac{1}{8} \|e^a \{e^{-a} \omega(y, x), f\}_{\star_\theta}\|^2 + \frac{1}{8} \|e^{-a} \{e^a \omega(y, x), f\}_{\star_\theta}\|^2.$$

So, the multiplier topology of \mathbf{B}_θ is equivalent to the Fréchet topology of $\mathcal{S}(\mathbb{S})$ and we have $\mathbf{B}_\theta = \mathcal{S}(\mathbb{S})$. ■

Remark 5.16 Theorem 5.15 extends to the framework of normal j -groups, or equivalently to Kählerian Lie groups with negative curvature. Such **normal j -groups** [26] are actually semidirect products of elementary normal j -groups

$$G = (\dots (\mathbb{S}_1 \ltimes_{\rho_1} \mathbb{S}_2) \ltimes_{\rho_2} \dots \mathbb{S}_{N-1}) \ltimes_{\rho_{N-1}} \mathbb{S}_N$$

where $\rho_i : \mathbb{S}_i \rightarrow \text{Aut}(\mathbb{S}_{i+1})$ are some (symplectic) actions. In [25], it has been proved that the star-products for G were tensor products of the elementary factors \mathbb{S}_i and that the modified Schwartz space was also a tensor product: $\mathcal{S}(G) := \mathcal{S}(\mathbb{S}_1) \hat{\otimes} \dots \hat{\otimes} \mathcal{S}(\mathbb{S}_N)$. Then, by taking the generators given in Theorem 5.15 for each factor \mathbb{S}_i , we obtain a symmetry of the HDQ $(L^2(G), \star_\theta)$ given by the tensor product of each $(L^2(\mathbb{S}_i), \star_\theta)$, and the Schwartz induced HDQ corresponds to the tensor product $\mathcal{S}(G)$. In the same way, results below concerning elementary normal j -groups \mathbb{S} can be extended to normal j -groups. ◆

Remark 5.17 Even if the kernel of the star-product (5.7) is different from the one of the Moyal-Weyl product (5.1), a slight adaptation of the proof of Theorem 5.9 leads to the same kind of result, namely that the unbounded multiplier space $\mathfrak{M}(\mathbf{B}_\theta)$, which is a complete nuclear locally convex Hausdorff $*$ -algebra by Proposition 3.34, identifies with the Fréchet multiplier space $\mathcal{M}_{\star_\theta}(\mathcal{S}(\mathbb{S}))$ defined and used for the star-exponential in [27]. ◆

Let us call the generators of \mathfrak{g} by H, y, y', E, E' (with $y, y' \in V$) such that the moment maps are

$$\eta_H = 2\ell, \quad \eta_y = e^{-a} \omega(y, x), \quad \eta_{y'} = e^a \omega(y', x), \quad \eta_E = e^{-2a}, \quad \eta_{E'} = e^{2a}.$$

Corollary 5.18 By Theorem 4.11, for any $T \in \mathfrak{g}$, the star-exponential $E_{\star_\theta}(\frac{i}{\theta} T)$ of the transvection symmetry \mathfrak{g} belongs to $\mathfrak{M}_{\text{stab}}(\mathbf{B}_\theta)$, so it is a unitary multiplier stabilizing $\mathbf{B}_\theta \simeq \mathcal{S}(\mathbb{S})$. We can find its explicit expression by using [27]. Namely, any arbitrary T in \mathfrak{g} writes

$$T(a, x, \ell) = \alpha \eta_H + \eta_y + \eta_{y'} + \beta \eta_E + \beta' \eta_{E'},$$

with $\alpha, \beta, \beta' \in \mathbb{R}$ and $y, y' \in V$. So, the star-exponential $E_{\star\theta}(\frac{i}{\theta}T)$ is the left and right \star_θ -multiplication by the function

$$(a, x, \ell) \mapsto \sqrt{\cosh(\alpha)} \cosh\left(\frac{\alpha}{2}\right)^n e^{\frac{i}{\theta} \sinh(\alpha)} \left(2\ell + \frac{\beta}{\alpha} e^{-2a} - \frac{\beta'}{\alpha} e^{2a} + \frac{e^{-a}}{\alpha} \omega(y, x) - \frac{e^a}{\alpha} \omega(y', x) \right).$$

Corollary 5.19 *The representation (5.9) restricted to \mathbf{B} is extendable, and it extends as a faithful \star -representation $\Omega : \mathfrak{M}(\mathbf{B}) \rightarrow \mathcal{L}^+(\mathcal{D}_{\mathbf{B}})$ of the multipliers of \mathbf{B} , with*

$$\mathcal{D}_{\mathbf{B}} := \{\varphi \in L^2(\mathbb{R}^{n+1}), \forall k_i \in \mathbb{N}, \forall \alpha_i \in \mathbb{N}^n, \sup_{a,v} |\sinh(2a)^{k_1} v^{\alpha_1} \partial_a^{k_2} \partial_v^{\alpha_2} \varphi(a, v)| < \infty\}.$$

PROOF Due to Theorem 4.9, it suffices to compute the image of Ω on generators of $\mathcal{U}(\mathfrak{g})$ to identify $\mathcal{D}_{\mathbf{B}}$. By using the explicit expression (5.9), we have

$$\begin{aligned} \Omega(\ell^k) \varphi(a_0, \ell_0) &= \left(\frac{i\theta}{4}\right)^k P_k(\partial_{a_0}) \varphi(a_0, v_0), & \Omega(\sinh(2a)^k) \varphi(a_0, v_0) &= \sinh(2a_0)^k \varphi(a_0, v_0), \\ \Omega(v^\alpha) \varphi(a_0, v_0) &= v_0^\alpha \varphi(a_0, v_0), & \Omega(w^\alpha) \varphi(a_0, v_0) &= \left(-\frac{i\theta}{2} \omega^{-1} \partial_{v_0}\right)^\alpha \varphi(a_0, v_0), \end{aligned}$$

where P_k is a polynomial of degree k . So the topology of $\mathcal{D}_{\mathbf{B}}$ is the exactly the one of the Corollary. \blacksquare

We can now give an explicit expression for the new Sobolev spaces associated to this symmetry.

Proposition 5.20 *The Sobolev spaces associated to the transvection symmetry are*

$$H^k(\mathbf{A}, \mathfrak{g}) = \{f \in L^2(\mathbb{S}), \forall X \in \mathcal{U}_k(\mathfrak{g}), \|X^* f\| < \infty\},$$

where $\mathcal{U}_k(\mathfrak{g})$ denotes the filtered enveloping algebra with generators of degree less or equal than k , fundamental vector fields are extended to elements of $\mathcal{U}(\mathfrak{g})$ by $(X_1 \dots X_p)^* = X_1^* \dots X_p^*$, and the fundamental vectors have the expression

$$\begin{aligned} H^* &= -\partial_a, & y^* &= -e^{-a} \partial_y + \frac{1}{2} e^{-a} \omega(x, y) \partial_\ell, & E^* &= -e^{-2a} \partial_\ell, \\ (y')^* &= -e^a \partial_{y'} - \frac{1}{2} e^a \omega(x, y') \partial_\ell, & (E')^* &= -e^{2a} \partial_\ell. \end{aligned}$$

Proposition 5.21 *Since \star_θ is strongly-invariant under the left-action of \mathbb{S} on itself, the moments maps (5.6) form another symmetry of the HDQ \mathbf{A} . However, the Schwartz induced HDQ does not correspond to the one of Theorem 5.15.*

PROOF The moment maps satisfy the relations:

$$[\eta_H, \eta_y]_{\star_\theta} = -i\theta \eta_y, \quad [\eta_H, \eta_E]_{\star_\theta} = -2i\theta \eta_E, \quad [\eta_y, \eta_{y'}]_{\star_\theta} = -i\theta \omega(y, y') \eta_E,$$

so they form a symmetry of the HDQ \mathbf{A} . Since the moment maps do not include the function e^{2a} , we see that the induced family of seminorms will correspond to a very different behavior from the ones of Theorem 5.15 for $a \rightarrow \infty$. \blacksquare

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